Lecture 12 - Polar Coordinates

Note Title

Ex: $x = \cos t$, $y = \sin t$, $-\pi/2 \le t \le \pi/2$, revolved around y-axis:

$$SA = \int_{-\pi/2}^{\pi/2} 2\pi x \, ds = 2\pi \int_{-\pi/2}^{\pi/2} 2\pi \cos t \sqrt{(\sin t)^2 + (\cos t)^2} \, dt = 2\pi \int_{-\pi/2}^{\pi/2} \cos t \, dt =$$

$$2\pi \sin t \Big|_{-\pi/2}^{\pi/2} = \boxed{4\pi}$$

Parametric area: Assume $x(t)$, $y(t)$ traces a curve once.

Then we can get areas:

$$A = \int_{x=a}^{x=b} y \, dx = \int_{\alpha}^{\beta} y(t)\, x'(t) \, dt$$

These are the t limits, but the order depends on which one gives a and b

Ex: $x = 3\cos t$ ← an ellipse: $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$y = 4 \sin t$ between $0 \le x \le 3$ (and above x-axis)

$$A = \int_0^3 y \, dx = \int_{\pi/2}^{0} (4 \sin t) \cdot (-3 \sin t) \, dt = \int_0^{\pi/2} 12 \sin^2 t \, dt$$

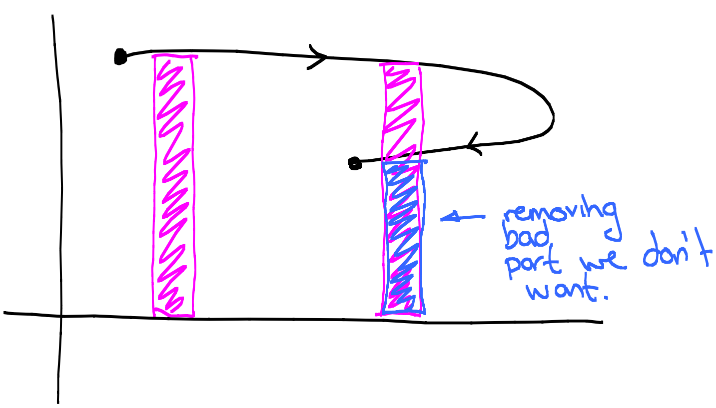
$x = 0 \Rightarrow t = \pi/2$

$x = 3 \Rightarrow t = 0$

$$= 12 \int_0^{\pi/2} \frac{1}{2} - \frac{1}{2} \cos 2t \, dt = \left(6t - \frac{1}{4} \sin 2t\right)\Big|_0^{\pi/2} = \boxed{3\pi}$$

Might worry about how dx and dt relate. Why have a $-$?

If the curve is not the graph of a function, then we have to cancel out parts we don't want to count

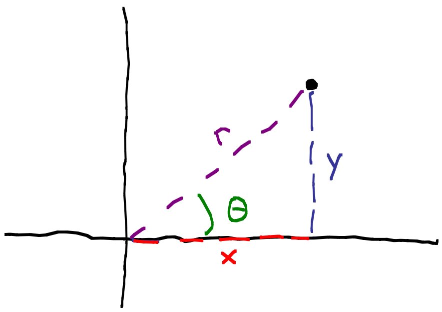


●: dx and dt both $+$: $+$ area

●: dx negative: $-$ area

Cartesian coord: to specify a point, give horizontal and vertical displacement from origin: (x,y)

Polar coord: to specify a point, you give direction, as an angle from x-axis, and how far to go: (r,θ)



r is a distance so $r \geq 0$
θ is an angle so $0 \leq \theta < 2\pi$

To do calculus, its helpful to relax these, so

$(-r,\theta) =$ "go r in dir opposite to θ"
$= (r, \theta+\pi)$

From picture, we get formulas relating x, y, r, θ:

$x = r\cos\theta$ $\qquad$ $r = \sqrt{x^2+y^2}$
$y = r\sin\theta$ $\qquad$ $\tan\theta = y/x$

Ex: What are the polar coordinates of $(1,\sqrt{3})$?

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$, $\quad \tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$, so $\boxed{(2, \pi/3)}$

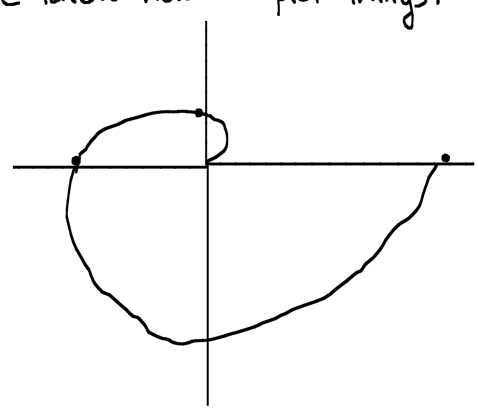
What are the rectangular coord of $(\sqrt{2}, 3\pi/4)$?

$$\left.\begin{aligned} x &= \sqrt{2}\cos\tfrac{3\pi}{4} = -1 \\ y &= \sqrt{2}\sin\tfrac{3\pi}{4} = 1 \end{aligned}\right\} \Rightarrow \boxed{(-1,1)}$$

Usually specific polar curves by expressing r in terms of θ:
$r = r(\theta)$

In this case, we know how to plot things: pick points and connect them.

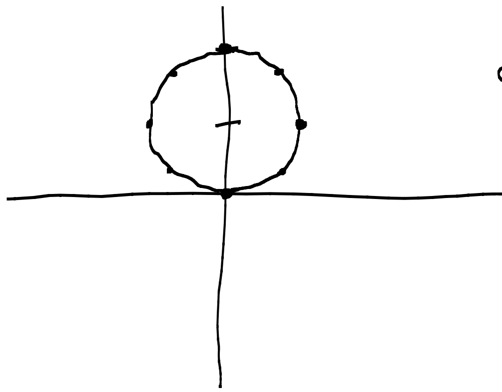
Ex: $\quad r = \theta$



a spiral!

Ex: $r = 2\sin\theta$:

θ	r
0	0
$\pi/6$	1
$\pi/4$	$\sqrt{2}$
$\pi/3$	$\sqrt{3}$
$\pi/2$	2
$2\pi/3$	$\sqrt{3}$
$3\pi/4$	$\sqrt{2}$
$5\pi/6$	1
π	0



a circle centered at $(0,1)$!

We can see this analytically:

$(r = 2\sin\theta)\cdot r \Rightarrow$

$$\underset{x^2+y^2}{\underset{\parallel}{r^2}} = 2\,\underset{y}{\underset{\parallel}{r\sin\theta}}$$

So $x^2+y^2-2y = 0$

or $x^2+y^2-2y+1 = 1$

$x^2+(y-1)^2 = 1$