Lecture 11 - Parametric Calculus & Polar

Note Title

Last time: If $x = x(t)$, $y = y(t)$, then

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

Ex: $x = \cos t$ $y = \sin t$ then

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = \boxed{-\cot t}$$

So the tangent line is horizontal at $t = \pi/2, 3\pi/2, 5\pi/2$, etc.

The slope is undefined at $t = 0, \pi, 2\pi$, etc $\rightsquigarrow$ here the tangent line is vertical!

Can repeat this procedure: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

This still tells us about concavity, etc.

Ex $x = t^2$ $y = t^3$ then

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$$

so $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{3}{2}t\right) \Big/ \dfrac{d}{dt}(t^2) = \boxed{\dfrac{3}{4t}}$

Arc length is also easier in this context.

$$dx = \frac{dx}{dt}\,dt \Rightarrow (dx)^2 = \left(\frac{dx}{dt}\right)^2 dt^2$$

$$dy = \frac{dy}{dt}\,dt \Rightarrow (dy)^2 = \left(\frac{dy}{dt}\right)^2 dt^2$$

So $ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 dt^2 + \left(\frac{dy}{dt}\right)^2 dt^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Ex: Find the arc length of

$x = 3\cos t$ $\quad 0 \le t \le \pi$

$y = 3\sin t$

$$\left(\frac{dx}{dt}\right)^2 = (-3\sin t)^2 = 9\sin^2 t \quad dt^2$$

$$+\left(\frac{dy}{dt}\right)^2 = (3\cos t)^2 = +9\cos^2 t \quad dt^2$$

$$ds^2 = 9\,dt^2$$

So $ds = 3\,dt \Rightarrow s = \int ds = \int_0^{\pi} 3\,dt = 3t\Big]_0^{\pi} = \boxed{3\pi}$

Ex: $x = 2t^2$, $y = t^3$ $\quad 0 \le t \le 1$.

$$\left(\frac{dx}{dt}\right)^2 = 16t^2\,dt^2$$

$$+\ \left(\frac{dy}{dt}\right)^2 = 9t^4\,dt^2$$

$$ds^2 = (16t^2 + 9t^4)\,dt^2$$

$$\Rightarrow ds = \sqrt{16t^2 + 9t^4}\,dt = t\sqrt{16 + 9t^2}\,dt$$

Arc length: $\displaystyle\int_0^1 t\sqrt{16+9t^2}\,dt$ $\qquad u = 16 + 9t^2$, $du = 18t\,dt$ $\qquad t = 0 \Rightarrow u = 16$, $t = 1 \Rightarrow u = 25$

$$= \frac{1}{18}\int_{16}^{25}\sqrt{u}\,du = \frac{2}{3}\cdot\frac{1}{18}u^{3/2}\Big]_{16}^{25} = \frac{1}{27}(125 - 64) = \boxed{\frac{61}{27}}$$

Having ds gives us surface area as well.

About x-axis: $\displaystyle SA = \int 2\pi y\,ds = 2\pi\int y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

About y-axis: $\displaystyle SA = \int 2\pi x\,ds = 2\pi\int x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Ex $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right\} \Rightarrow \quad ds = \sqrt{(4\sin t)^2 + (4\cos t)^2}\,dt = \sqrt{4}\,dt = 2\,dt$

$-\pi/2 \le t \le \pi/2$

surface is a sphere of radius 2.

Revolved about y-axis.

$$SA = \int_{-\pi/2}^{\pi/2} 2\pi\cdot 2\cos t\cdot 2\,dt = 8\pi\int_{-\pi/2}^{\pi/2}\cos t\,dt = 8\pi\sin t\Big]_{-\pi/2}^{\pi/2} = \boxed{16\pi}$$

Ex $\left.\begin{array}{l} x = t^2 \\ y = t^4 \end{array}\right.$ $\quad 0 \le t \le 1$ $\Big\} \Rightarrow ds = \sqrt{4t^2 + 16t^6}\,dt = 2t\sqrt{1 + 4t^4}\,dt$

revolved about y-axis $\Rightarrow S = \int 2\pi x\,ds$

$$S = \int_0^1 2\pi t^2\cdot 2t\sqrt{1 + 4t^4}\,dt = \int_0^1 4\pi t^3\sqrt{1+4t^4}\,dt$$

$u = 1 + 4t^4$, $du = 16t^3\,dt$ $\qquad t = 0 \Rightarrow u = 1$, $t = 1 \Rightarrow u = 5$

$$S = \int_1^5 \frac{\pi}{4}\sqrt{u}\,du = \frac{2}{3}\frac{\pi}{4} u^{3/2}\Big|_1^5 = \boxed{\frac{\pi}{6}\left(5\sqrt{5}-1\right)}$$

Lastly, $x = x(t)$, $y = y(t)$ is **smooth** at $t = a$ if $x'(a)$ and $y'(a)$ are not both 0.

Ex $x = \cos t$, $y = \sin t$ is smooth everywhere

$x = t^2$, $y = t^3$ is not smooth at $t = 0$: