MATH 132 FINAL 12/11/07 3-hour closed-book

NO CALCULATORS; SHOW **ALL** WORK; Put ANSWERS in BOXES

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Name : Solutions SCORE /300 = ______

Missed p1____ p2 ____ p3____ p4____ p5____ p6____ p7____ p8____ Total missed ______

Problem 1: Find the following definite and indefinite integrals:

(a) $\int_0^1 \ln(1+x^2)\,dx = \boxed{\ln(2) - 2 + \pi/2}$

$\int \ln(1+x^2)\,dx$ let $u = \ln(1+x^2)$ & $dv = dx$

$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$ $\Rightarrow du = \frac{2x}{1+x^2}$ $v = x$

$= x\ln(1+x^2) - \int 2 - \frac{2}{1+x^2}\,dx \leftarrow$

$= x\ln(1+x^2) - 2x + 2\arctan(x) + C$

$\Rightarrow \int_0^1 \ln(1+x^2)dx = \ln(2) - 2 + 2(\pi/4) - 0 + 0 - 0$

Long Division

$$\begin{array}{r} 2 \\ x^2+0x+1 \,\overline{)\,2x^2+0x+0} \\ \underline{2x^2+0x+2} \\ -2 \end{array}$$

$\Rightarrow \frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x2}$

(b) $\int \sin^2(x)\cos^3(x)\,dx = \boxed{\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x)} + C$

$= \int \sin^2(x)\cos^2(x)\cos(x)\,dx$

$= \int \sin^2(x)(1-\sin^2(x))\cos(x)dx$ let $u = \sin(x)$ $\Rightarrow du = \cos(x)\,dx$

$= \int u^2(1-u^2)\,du$

$= \int u^2 - u^4\,du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5$

(c) $\int \frac{dx}{(a^2-x^2)^{3/2}} = \boxed{\frac{1}{a^2}\;\frac{x}{\sqrt{a^2-x^2}}} + C$

$= \int \frac{a\cos\theta}{a^3\cos^3\theta}\,d\theta$

let $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\,d\theta$

$(a^2-x^2)^{3/2} = (a^2 - a^2\sin^2\theta)^{3/2} = (a^2\cos^2\theta)^{3/2} = a^3\cos^3\theta$

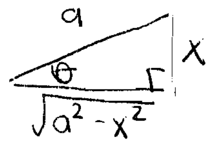
$= \frac{1}{a^2}\int \sec^2\theta\,d\theta$

$= \frac{1}{a^2}\tan\theta$

$\sin\theta = \frac{x}{a}$

$= \frac{1}{a^2}\;\frac{x}{\sqrt{a^2-x^2}} + C$

$\Rightarrow \tan\theta = \frac{x}{\sqrt{a^2-x^2}}$



Problem 2: Write the **form** of the partial fraction decomposition of the following rational functions (DO NOT SOLVE FOR THE CONSTANTS):

(a) $\frac{2x+3}{x(x^2+2x+1)} = \boxed{\frac{A}{X} + \frac{B}{X+1} + \frac{C}{(X+1)^2}}$

$\overset{\shortparallel}{(X+1)^2}$

(b) $\frac{2x+3}{x(x^2+2x+2)} = \boxed{\frac{A}{X} + \frac{BX+C}{X^2+2X+2}}$

Problem 3: (a) If f is a continuous function on $(-\infty, a]$, then the integral $\int_{-\infty}^{a} f(x)dx$ is **defined** to be convergent if:

$\lim_{t \to -\infty} \int_t^a f(x)\,dX$ exists

(b) Determine whether the improper integral $\displaystyle\int_1^\infty \frac{e^{\sin(x)}}{x^2+3}\,dx$ converges; explain your reasoning, but **do not evaluate** the integral if it converges.

$-1 \le \sin X \le 1 \Rightarrow 1/e \le e^{\sin X} \le e$

converges ☑ diverges ☐

Then, $0 \le \int_1^\infty \frac{e^{\sin(X)}}{X^2+3}\,dX \le \int_1^\infty \frac{e}{X^2+3}\,dX \le \int_1^\infty \frac{e}{X^2}\,dX \leftarrow$ converges

$\Rightarrow \int_1^\infty \frac{e^{\sin(X)}}{X^2+3}\,dX$ converges by comparison

$2X-3=0$
$2X=3$
$X=3/2$

(c) Evaluate the improper integral $\displaystyle\int_1^4 \frac{1}{2x-3}\,dx$ (write DIV if it does not converge). Explain your reasoning.

$\lim_{t \to 3/2^-} \int_1^t \frac{1}{2X-3}\,dX + \lim_{t \to 3/2^+} \int_t^4 \frac{1}{2X-3}\,dX$

$\displaystyle\int_1^4 \frac{1}{2x-3}\,dx = \boxed{\text{DIV}}$

$= \lim_{t \to 3/2^-} \frac{1}{2}\ln|2X-3|\Big|_1^t + \lim_{t \to 3/2^+} \frac{1}{2}\ln|2X-3|\Big|_t^4$

$= \lim_{t \to 3/2^-} \frac{1}{2}\ln|2t-3| - \frac{1}{2}\cancel{\ln}1 + \lim_{t \to 3/2^+} \frac{1}{2}\ln(5) - \underset{\to -\infty}{\frac{1}{2}\ln|2t-3|}$

Problem 4: Answer the following questions for the **parameterized curve** $x = e^t, y = e^{-t}$ for $0 \leq t \leq 1$; explain your reasoning.

(a) Set up (but **do not evaluate**) an integral for the length of the curve.

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \boxed{\int_0^1 \sqrt{(e^t)^2 + (e^{-t})^2}\, dt}$$

$$\frac{dx}{dt} = e^t, \quad dy/dt = -e^{-t}$$

(b) Find the equation of the tangent line to the curve at time $t = \frac{1}{2}$.

$$m_{tan} = \frac{dy/dt}{dx/dt} = \frac{-e^{-t}}{e^t} = \frac{-1}{e^{2t}}$$

$$y = \boxed{-1/e}\ x + \boxed{2/\sqrt{e}}$$

$$\Rightarrow m_{tan}\big|_{t=1/2} = -1/e$$

when $t = 1/2$, $x = e^{1/2}$, $y = e^{-1/2}$

$$y - e^{-1/2} = -1/e\,(x - e^{1/2})$$

$$\Rightarrow y = -\frac{1}{e}x + \frac{\sqrt{e}}{e} + \frac{1}{\sqrt{e}}$$

$$= -\frac{1}{e}x + \frac{\sqrt{e} + \sqrt{e}}{e} = -\frac{1}{e}x + \frac{2\sqrt{e}}{e} = \boxed{-\frac{1}{e}x + \frac{2}{\sqrt{e}}}$$

(c) Find the area between the curve and the x-axis from $t = 0$ to $t = 1$;

$$A = \int_0^1 y\, dx$$

$$Area = \boxed{1}$$

$$= \int_0^1 e^{-t} \cdot e^t\, dt$$

$$= \int_0^1 1\, dt$$

$$= t\big|_0^1$$

$$= 1$$

(d) Find a Cartesian equation $y = f(x)$ for this curve.

$$x = e^t$$

$$\Rightarrow \ln x = t \qquad \Rightarrow y = e^{-\ln x}$$

$$= e^{\ln(x^{-1})}$$

$$= \frac{1}{x}$$

$$y = \boxed{1/x}$$

Problem 5: (a) Solve the differential equation $y' + y = \frac{1}{1+e^x}$ with initial condition $y(0) = 0$.

$y(x) =$

omit

(b) Find the most general solution of $\frac{dy}{dx} = 5yx^4$.

$y(x) =$

omit

Problem 6: (a) Find the area of the wedge-shaped region inside the polar curve $r = \cos(\theta)$ between $\theta = 0$ and $\theta = \frac{\pi}{4}$.

$$A = \int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \cos^2\theta \, d\theta$$

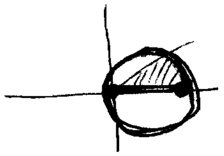
$$= \frac{1}{4} \int_0^{\pi/4} 1 + \cos(2\theta) \, d\theta$$

$$= \frac{1}{4} \left[\theta + \frac{1}{2} \sin(2\theta) \Big|_0^{\pi/4} \right]$$

$$= \frac{1}{4} \left[\pi/4 + \frac{1}{2} \sin(\pi/2) - 0 - 0 \right]$$

$$= \frac{\pi}{16} + 1/8$$

Area = $\boxed{\frac{\pi}{16} + 1/8}$



(b) Give the Cartesian equation of the polar curve $r = \frac{1}{\cos(\theta) + \sin(\theta)}$.

$$\left[r = \frac{1}{\cos\theta + \sin\theta} \right]$$

$$\Rightarrow \quad r(\cos\theta + \sin\theta) = 1$$

$$\Rightarrow \quad r\cos\theta + r\sin\theta = 1$$

$$\Rightarrow \quad x + y = 1$$

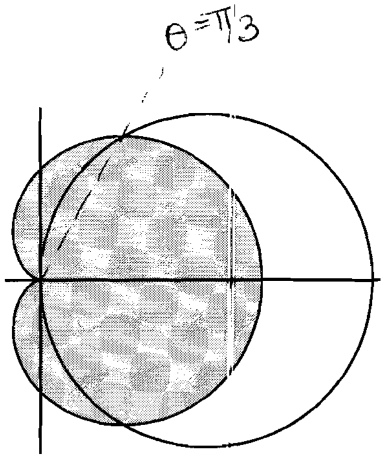
$\boxed{x + y = 1}$

(c) For the circle $r = 3\cos(\theta)$ and the cardioid $r = 1 + \cos(\theta)$, set up (but **do not evaluate**) the integral for the area inside the circle but outside the cardioid (the **unshaded** crescent region sketched below):

Intersection Points: $3\cos\theta = 1+\cos\theta$

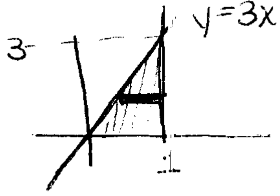
$\cos\theta = 1/2 \Rightarrow \theta = \pi/3, -\pi/3$

$$A = \int_{-\frac{\pi}{3}}^{\pi/3} \frac{1}{2}(3\cos\theta)^2 - \frac{1}{2}(1+\cos\theta)^2 \, d\theta$$



Area = $\boxed{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(3\cos\theta)^2 - \frac{1}{2}(1+\cos\theta)^2 d\theta}$

Problem 7: Write the iterated integral $\int_0^1 \int_0^{3x} f(x,y)dydx$ as an iterated integral with the order of integration interchanged.



$y = 3x \Leftrightarrow \frac{1}{3}y = x$

$$\boxed{\int_0^3 \int_{\frac{1}{3}y}^{1} f(x,y)\, dx\, dy}$$

Problem 8: In the following 3 parts, state the definitions of convergence, absolute convergence, and conditional convergence of an infinite series $\sum_{n=1}^{\infty} a_n$. The series $\sum_1^{\infty} a_n$ is **defined** to be

(a) **convergent** if: $\lim_{n\to\infty} S_n$ exists, $S_n = a_1 + a_2 + \cdots + a_n$

(b) **absolutely convergent** if: $\sum_{n=1}^{\infty} |a_n|$ converges

(c) **conditionally convergent** if the series converges, but not absolutely

Problem 9: Determine whether the following infinite series converge absolutely, conditionally, or diverge (check one). Mention which tests you use.

(a) $\sum_{n=1}^{\infty} \ln\left(\frac{2n^2+1}{n^2+3}\right)$ converges absolutely ☐ converges conditionally ☐ diverges ☑

$\lim_{n \to \infty} \ln\left(\frac{2n^2+1}{n^2+3}\right) = \ln(2) \neq 0$

$\Rightarrow$ diverges by test for divergence

(b) $\sum_{n=1}^{\infty} \frac{2+\cos(n)}{\sqrt{n}}$ converges absolutely ☐ converges conditionally ☐ diverges ☑

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{2+\cos(n)}{\sqrt{n}}$ diverges by comparison to p-series

Problem 10: Find **all** values of q such that the following infinite series converge: $(p=-q)$

(a) $\sum_{n=1}^{\infty} n^q$ converges all q with $\boxed{q<-1}$ $\leftarrow$ p-test $\sum \frac{1}{n^p} < \infty$ for $p>1$

(b) $\sum_{n=1}^{\infty} (-1)^n n^q$ converges for all q with $\boxed{q<0}$ $\leftarrow$ alt. series test

(c) $\sum_{n=1}^{\infty} q^n$ converges for all q with $\boxed{|q|<1}$ $\leftarrow$ geometric series

(d)Find the **interval of convergence** of $\sum_{n=0}^{\infty} \frac{n!(x+5)^n}{4^n(n+1)!}$. Interval $\boxed{-9 \leq x < -1}$

$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left| \frac{(n+1)!(x+5)^{n+1}}{4^{n+1}(n+2)!} \cdot \frac{4^n(n+1)!}{n!(x+5)^n} \right|$

$= \lim_{n \to \infty} \frac{1}{4} \cdot \frac{1}{n+2} \cdot \frac{n+1}{1} \cdot |x+5| = \frac{|x+5|}{4} < 1 \Leftrightarrow -4 < x+5 < 4$

$\Leftrightarrow -9 < x < -1$

check endpoints:

$\underline{x=-1}$: $\sum \frac{4^n}{4^n(n+1)}$ div. by p-test

$\underline{x=-9}$: $\sum (-1)^n \frac{1}{n+1}$ conv. by AST

Problem 11. (a) For a general differentiable function f, the **Taylor series** centered at a is defined to be $\sum_{n=0}^{\infty} c_n(x-a)^n$ where the coefficients are

$$c_n = \boxed{\frac{f^{(n)}(a)}{n!}}$$

(b) Find the Taylor series for the function $f(x) = e^x$ centered at $\ln(5)$.

$f^{(n)}(x) = e^x$, $f^{(n)}(\ln 5) = 5$, $a_n = 5/n!$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{5}{n!}(x - \ln 5)^n}$$

(c) Find the fourth-degree Taylor polynomial $T_4(x)$ centered at 0 for the function $f(x) = \int_0^x e^{-t^2}\,dt$

$T_4(x) = \sum_{n=0}^{4} c_n x^n$ for coefficients $c_0 = \boxed{0}$ $c_1 = \boxed{1}$ $c_2 = \boxed{0}$ $c_3 = \boxed{-\frac{1}{3}}$ $c_4 = \boxed{0}$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$\Rightarrow f(x) = \int_0^x e^{-t^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \dots$$

Problem 12: Identify the following power series with the indicated functions (put the letter of the correct function next to the series):

D (i) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = e^{-x}$

C (ii) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

F (iii) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$

A (iv) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

A: $\arctan(x)$
B: $\arctan(-x)$
C: $-\ln(1+x)$
D: $\frac{1}{e^x}$
E: $-e^x$
F: $\frac{\sin(x)}{x}$
G: $\sin(x)$
H: $\frac{\cos(x)}{2n+1}$

$\frac{1}{1+x} = \sum_{0}^{\infty} (-1)^n x^n$, $\ln(1+x) = \sum_{0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$, $\frac{1}{1+x^2} = \sum_{0}^{\infty} (-1)^n x^{2n}$

Problem 13: Find the sum of the following series:

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{6^{2n+1}(2n)!} = \boxed{\frac{\pi^2\sqrt{3}}{12}}$

$= \frac{\pi^2}{6} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$

$= \frac{\pi^2}{6} \cos\left(\frac{\pi}{6}\right) = \frac{\pi^2}{6} \cdot \frac{\sqrt{3}}{2}$

(b) $-\ln(2) + \frac{(\ln(2))^2}{2!} - \frac{(\ln(2))^3}{3!} + \ldots = \boxed{-1/2}$

$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = e^{-x}$

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{(\ln(2))^n}{n!} = e^{-\ln(2)} = \frac{1}{2}$ & $\sum_{n=1}^{\infty} (-1)^n \frac{(\ln 2)^n}{n!} = \frac{1}{2} - \underset{\substack{\uparrow \\ \text{"}n=0\text{"}}}{1} = -\frac{1}{2}$

True-False Questions

Each is worth 4 points; circle the correct answer T or F (you do not need to justify your answer).

1. T [✓ F] If $\lim_{b\to\infty} \int_{-b}^{b} f(x)dx$ exists, then $\int_{-\infty}^{\infty} f(x)dx$ converges.

2. [✓ T] F The graph of the parametric curve $x = 2t^{3/5}$, $y = t^{3/5}$ is a straight line.

   $\frac{x}{2} = t^{3/5} \Rightarrow y = \frac{x}{2}$

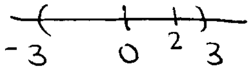
3. [✓ T] F If $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ must converge.

   (absolute conv.

4. [✓ T] F Every power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for at least one value of x.

   $x = \{a\}$

5. [✓ T] F If the series $\sum_{1}^{\infty} \underset{\substack{\uparrow \\ a=0}}{c_n x^n}$ converges for all $-3 < x < -1$, then $\sum_{1}^{\infty} c_n 2^n$ must converge as well.



PLEDGE IN FULL: