We have almost proved our main theorem. Only existence of covering spaces remains.

<u>Def</u> A path connected covering space $\tilde{X} \xrightarrow{p} X$ s.t. $\pi_1(\tilde{X}, \tilde{x}) = \{e\}$ is a <u>universal covering space</u>.

<u>Prop</u> If $\tilde{X}$ exists, it is unique.

Pf: This is immediate from the uniqueness results from last time.

<u>Thm</u> If X is p.c. & $\forall x \in X$ & $U \ni x$, $\exists\ x \in V \subseteq U$ s.t. $\pi_1(V,x) \to \pi_1(U,x)$ is trivial, then a universal cover exists.

Pf: Below.

The crazy part about the universal cover is that $\pi_1(X,x)$ acts on it.

<u>Def</u> Let G be a group and X a set or space. A G-action on X is a map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g,x) &\longmapsto g\cdot x \end{aligned} \qquad \text{s.t.} \quad ① \ e\cdot x = x \ \ \forall x \in X \ \ \& \quad ② \ g\cdot(h\cdot x) = (g\cdot h)\cdot x \ \ \forall g,h \in G,\ x \in X.$$

<u>Prop</u>: Let $\tilde{X} \xrightarrow{p} X$ be a covering map & let $x \in X$. The assignment

$$\begin{aligned} \pi_1(X,x) \times p^{-1}(x) &\longrightarrow p^{-1}(x) \\ (\gamma, \tilde{x}) &\longmapsto \tilde{\gamma}_{\tilde{x}}(1), \end{aligned}$$

where $\tilde{\gamma}_{\tilde{x}}$ is the lift of γ starting at $\tilde{x}$ gives a G-action.

Pf: If $[\gamma]$ is $[c_x]$ then $[c_{\tilde{x}}] = [\tilde{\gamma}_{\tilde{x}}]$ for all $\tilde{x} \in p^{-1}(x)$, so $\tilde{\gamma}_{\tilde{x}}(1) = \tilde{x}$.

Now consider $([\gamma_1]\cdot[\gamma_2])\cdot\tilde{x}$. This is a lift of $\gamma_1 * \gamma_2$, starting at $\tilde{x}$, and then evaluated at 1: $\widetilde{\gamma_1 * \gamma_2}_{\tilde{x}}(1)$. Now $\tilde{\gamma}_{1\,\tilde{\gamma}_{2\tilde{x}}(1)} * \tilde{\gamma}_{2\tilde{x}}$ is also a lift of $\gamma_1 * \gamma_2$, starting at $\tilde{x}$, so by unique lifting, $\widetilde{\gamma_1 * \gamma_2}_{\tilde{x}} = \tilde{\gamma}_{1\,\tilde{\gamma}_{2\tilde{x}}(1)} * \tilde{\gamma}_{2\tilde{x}}$. Unpacking this, we see $\widetilde{\gamma_1 * \gamma_2}_{\tilde{x}}(1) = \gamma_1 \cdot (\gamma_2 \cdot \tilde{x})$. $\square$

<u>Prop</u> If $\tilde{X}$ is connected, then $\forall\, \tilde{x} \in p^{-1}(x)$, $\begin{aligned} \pi_1(X,x) &\longrightarrow p^{-1}(x) \\ \gamma &\longmapsto \gamma\cdot\tilde{x} \end{aligned}$ is a surjection.

Pf: Let $\tilde{x}_1 \in p^{-1}(x)$ and choose a path $\tilde{x} \xrightarrow{\tilde{\gamma}} \tilde{x}_1$. By construction, $\gamma = p \circ \tilde{\gamma}$ is in $\pi_1(X,x)$ & $\gamma\cdot\tilde{x} = \tilde{x}_1$. $\square$

<u>Prop</u> $\gamma_1 \cdot \tilde{x} = \gamma_2 \cdot \tilde{x}$ iff $\gamma_1^{-1}\cdot\gamma_2 \in p(\pi_1(\tilde{X},\tilde{x}))$.

Pf: Multiplying both sides by γ_1^{-1} shows it suffices to prove "$\gamma\cdot\tilde{x} = \tilde{x}$ iff $\gamma \in p(\pi_1(\tilde{x},\tilde{x}))$." Now $\gamma\cdot\tilde{x} = \tilde{x}$ iff γ lifts to a loop starting at $\tilde{x}$ iff $\gamma \in p_*(\pi_1(\tilde{X},\tilde{x}))$. $\square$

<u>Cor</u>: If $\tilde{X}$ is the universal cover, then $\begin{aligned} \pi_1(X,x) &\longrightarrow p^{-1}(x) \\ \gamma &\longmapsto \gamma\cdot\tilde{x} \end{aligned}$ is a bijection.

Now we can put this all together.

<u>Def</u> Let $\tilde{X} \xrightarrow{p} X$ be the universal cover & fix $\tilde{x} \in X$. Then for each $\gamma \in \pi_1(X,x)$, let

$F_\gamma : \tilde{X} \longrightarrow \tilde{X}$ be the lift of p over p sending $\tilde{x}$ to $\gamma^{-1} \cdot \tilde{x}$.

<u>Prop</u>: $F_\gamma(h \cdot \tilde{x}) = h \cdot F_\gamma(\tilde{x}) \quad \forall h, \gamma \in \pi_1(X,x)$.

<u>Pf</u>: $h \cdot \tilde{x}$ is computed by lifting h to a path $\tilde{h}_{\tilde{x}}$ starting at $\tilde{x}$ and then evaluating at 1.

So $F_\gamma(h \cdot \tilde{x}) = F_\gamma \circ (\tilde{h}_{\tilde{x}})(1)$. $F_\gamma \circ \tilde{h}_{\tilde{x}}$ is a path starting at $F_\gamma \circ \tilde{h}_{\tilde{x}}(0) = \gamma^{-1} \cdot \tilde{x}$ and lifting

$h. \Rightarrow F_\gamma \circ \tilde{h}_{\tilde{x}} = \tilde{h}_{\gamma^{-1} \cdot \tilde{x}}$ & hence $F_\gamma \circ \tilde{h}_{\tilde{x}}(1) = h \cdot (\gamma^{-1} \tilde{x})$. $\square$

<u>Prop</u>:
$$\begin{aligned} \pi_1(X,x) \times \tilde{X} &\longrightarrow \tilde{X} \\ (\gamma, y) &\longmapsto F_\gamma(y) \end{aligned}$$
is a $\pi_1(X,x)$ action.

<u>Pf</u>: First, since $e \cdot \tilde{x} = \tilde{x}$, $F_e : \tilde{X} \to \tilde{X}$ is a lift fixing a point, hence is the identity.

Second $F_{\gamma_1} \circ F_{\gamma_2}$ and $F_{\gamma_1 * \gamma_2}$ are two lifts & they agree on $\tilde{x}$:

$F_{\gamma_1}(F_{\gamma_2}(\tilde{x})) = F_{\gamma_1}(\gamma_2^{-1} \cdot \tilde{x}) = \gamma_2^{-1} \cdot F_{\gamma_1}(\tilde{x}) = \gamma_2^{-1} \cdot (\gamma_1^{-1} \cdot \tilde{x}) = (\gamma_1 \cdot \gamma_2)^{-1} \cdot \tilde{x} = F_{\gamma_1 \cdot \gamma_2}(\tilde{x})$. So $F_{\gamma_1} \circ F_{\gamma_2} = F_{\gamma_1 \cdot \gamma_2}$ everywhere. $\square$

<u>Cor</u> If $U \subseteq X$ is evenly covered, then a choice of point $\tilde{x} \in p^{-1}(U)$ gives an iso of G-spaces
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \in \tilde{U}$"

$$\begin{aligned} G \times U &\longrightarrow p^{-1}(U) \text{ over } U. \\ (g,u) &\longmapsto g \cdot p|_{\tilde{U}}^{-1}(u) \end{aligned}$$

<u>Def</u> If $H \subseteq \pi_1(X,x)$, then let $\sim_H$ on $\tilde{X}$ be $x \sim_H y$ iff $\exists\, h \in H$ $y = h \cdot x$.

<u>Prop</u> $\sim_H$ is an equivalence relation.

<u>Pf</u>: $e \in H \Rightarrow x \sim_H x$. $h \in H \Rightarrow h^{-1} \in H \Rightarrow (y = h \cdot x \Rightarrow x = h^{-1} \cdot y)$. $y = h_1 \cdot x$, $z = h_2 \cdot y \Rightarrow z = (h_2 \cdot h_1) \cdot x$. $\square$

<u>Def</u> Let $\tilde{X}_H = \tilde{X}/\sim_H$. Let $p_H : \tilde{X}_H \longrightarrow X$ be the quotient map.

Since $p^{-1}(U) \cong G \times U$ as G-spaces, the quotient is just "identifying pancakes" & hence is locally trivial.

<u>Thm</u> Let $\tilde{x} \in \tilde{X}$ be the point used to define $F_{\cdot}$... Then $p_* \pi_1(\tilde{X}_H, [\tilde{x}]) = H$.

<u>Pf</u>: γ in $\pi_1(X,x)$ is in $p_* \pi_1(\tilde{X}_H, [\tilde{x}])$ iff $\gamma \cdot [\tilde{x}] = [\gamma \cdot \tilde{x}] = [\tilde{x}]$ iff $\gamma \in H$. $\square$

The only thing we haven't shown is that (unpointed covering spaces) $\longleftrightarrow$ (conjugacy classes of s.g. of π_1)

This is just the "change of basepoint."

<u>Prop</u>: If $\tilde{x}_0$ & $\tilde{x}_1 \in p^{-1}(x)$, then $p_*(\pi_1(\tilde{X},\tilde{x}_0)) = \gamma\, p_*(\pi_1(\tilde{X},\tilde{x}_1))\, \gamma^{-1}$, where $\gamma = p \circ \tilde{\gamma}$, $\tilde{\gamma}$ a path $\tilde{x}_1 \to \tilde{x}_0$.