Prop: Let $p: \tilde{X} \to X$ be a covering map with $\tilde{X}$ & X path connected. Then for any $\tilde{x} \in \tilde{X}$, if $x = p(\tilde{x})$, then $p_*: \pi_1(\tilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x)$ is injective. The image is those loops in $\pi_1(X, x)$ that lift to loops in $\tilde{X}$ starting at $\tilde{x}$.

Pf: Let $\tilde{\gamma}$ be a loop in $\tilde{X}$ based at $\tilde{x}$, and let $\gamma = p \circ \tilde{\gamma}$ (i.e. $p_*([\tilde{\gamma}])$). If $\gamma \simeq_{[0,1]} c_x$, then we can lift the homotopy rel endpoints to one $\tilde{\gamma} \simeq_{[0,1]} c_{\tilde{x}}$. In other words $p_*([\tilde{\gamma}]) = e$ $\Rightarrow [\tilde{\gamma}] = e$. Now if $p_*(\tilde{\gamma}_1) = p_*(\tilde{\gamma}_2)$, then $p_*([\tilde{\gamma}_1][\tilde{\gamma}_2]^{-1}) = e \Rightarrow [\tilde{\gamma}_1] \cdot [\tilde{\gamma}_2]^{-1} = e \Rightarrow [\tilde{\gamma}_1] = [\tilde{\gamma}_2]$.

For the second part, by definition, if $\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x})$, then $[p \circ \tilde{\gamma}]$ lifts to a loop starting at $\tilde{x}$ ($\tilde{\gamma}$). Conversely, if a loop γ lifts to a loop $\tilde{\gamma}$, then $[\gamma] = p_*([\tilde{\gamma}])$. □

We can do (much) better.

Def A space is <u>locally path connected</u> if $\forall x \in X$ & $U \ni x$ open, there is a path connected neighborhood $x \in V \subseteq U$.

Thm Let Y be path conn. & locally path conn. Let $\tilde{X} \xrightarrow{p} X$ be a covering map, and let $f: Y \to X$. Let $y \in Y$, $\tilde{x} \in \tilde{X}$ have $f(y) = p(\tilde{x}) =: x$. Then there is a lift $\tilde{f}: Y \to \tilde{X}$ with $y \mapsto \tilde{x}$ iff

$$f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x})).$$

Pf: $\Rightarrow$) If such a lift exists, then $f_* = p_* \circ \tilde{f}_*$, and hence $\mathrm{Im}(f_*) \subseteq \mathrm{Im}(p_*)$.

$\Leftarrow$) Since Y is path connected, given any $y' \in Y$, we can find a path $\gamma_{y'}$ starting at y and ending at y'. Let $\tilde{f}(y') = \widetilde{(f \circ \gamma_{y'})}(1)$, where we start our lift at $\tilde{x}$.

① $\tilde{f}$ is well-defined: Let $\gamma_{y'}$ and $\gamma'_{y'}$ be two paths $y \to y'$. Then $\overline{\gamma'_{y'}} * \gamma_{y'}$ is a loop starting at y, so $f \circ (\overline{\gamma'_{y'}} * \gamma_{y'})$ is in $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}))$. This means that it lifts to a loop based at $\tilde{x}$: $\widetilde{f \circ (\overline{\gamma'_{y'}} * \gamma_{y'})}$. By unique path lifting, this is $\overline{\widetilde{f \circ \gamma'_{y'}}} * \widetilde{f \circ \gamma_{y'}}$, and hence $\widetilde{f \circ \gamma'_{y'}}(1) = \widetilde{f \circ \gamma_{y'}}(1)$. So $\tilde{f}$ is independent of the path.

② $\tilde{f}$ is continuous: Let $y' \in Y$ & let $\tilde{f}(y) \in \tilde{U} \subseteq X$, $\tilde{U}$ s.t. (i) $p(\tilde{U}) = U$ is open & (ii) $p|_{\tilde{U}}: \tilde{U} \xrightarrow{\cong} U$. We must find a V s.t. $\tilde{f}(V) \subseteq \tilde{U}$. Let $W = f^{-1}(U)$. Since f is continuous, $f^{-1}(U) \ni y'$ is open. By local path connectedness, there is a $V \ni y'$, $V \subseteq f^{-1}(U)$ s.t. V is p.c.

Now fix a path $y \xrightarrow{\gamma_{y'}} y'$. For each $y_0 \in V$, let γ_0 be a path in V from $y' \longrightarrow y_0$. Then $\gamma_0 * \gamma_{y'}$ is a path $y \to y_0$, and hence can be used to find $\tilde{f}(y_0)$. But since $f(V) \subseteq U$ is evenly covered, the lifts of $f\circ\gamma_0$ are necessarily $p|_{\tilde U}^{-1}\circ(f\circ\gamma_0)$. In particular, $p|_{\tilde U}^{-1}\circ f\circ\gamma_0(1)$ just becomes $p|_{\tilde U}^{-1}\circ f(\gamma_0(1))$, and

$\tilde{f}(y_0) = p|_{\tilde U}^{-1}\circ f(y_0)$. □

Cor Let $\tilde{X}_i \xrightarrow{p_i} X$, $i=1,2$ be covering maps with $\tilde{X}_i$ & X p.c. & locally p.c. If $\tilde{x}_1$ and $\tilde{x}_2$ have $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$, then there is a homeomorphism $\tilde{X}_1 \xrightarrow{f_{12}} \tilde{X}_2$ over X iff $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2,\tilde{x}_2))$

$\tilde{x}_1 \longmapsto \tilde{x}_2$ $p_2\circ f_{12} = p_1$.

Pf: ⇒) This is ⇒) of the previous lemma.

⇐) By the previous lemma, we have lifts:

$$\begin{tikzcd} & \tilde{X}_2 \arrow[d, "p_2"] \\ \tilde{X}_1 \arrow[ur, "f_{12}"] \arrow[r, "p_1"] & X \end{tikzcd} \quad \& \quad \begin{tikzcd} & \tilde{X}_1 \arrow[d, "p_1"] \\ \tilde{X}_2 \arrow[ur, "f_{21}"] \arrow[r, "p_2"] & X \end{tikzcd}.$$

Now consider $f_{12}\circ f_{21} : \tilde{X}_2 \longrightarrow \tilde{X}_2$. By construction, this is a lift of $X_2 \longrightarrow X$ & $f_{12}\circ f_{21}(\tilde{x}_2) = \tilde{x}_2$. The identity on $\tilde{X}_2$ is another lift with these properties so $f_{12}\circ f_{21} = \mathrm{Id}$. Switching 1 & 2 ⇒ $f_{21}\circ f_{12} = \mathrm{Id}$ &

$\tilde{X}_1 \cong \tilde{X}_2$. □

In words: p.c. locally p.c. covering spaces are completely determined by which s.g. of $\pi_1(X,x)$ we see.