Thm (Homotopy lifting) Given $F: Y\times I \to X$, $p: \tilde{X} \to X$ a covering map, & $\tilde{f}: Y\times\{0\} \to \tilde{X}$ lifting $F|_{Y\times\{0\}}$, there is a unique $\tilde{F}: Y\times I \to \tilde{X}$ lifting F.

$p^{-1}(u) = \coprod U_i \ldots$

Pf: Let $y \in Y$, and consider $\{y\}\times I \subseteq Y\times I$. For each $t\in I$, choose an evenly covered open set $U_{F(y,t)} \subseteq X$ containing $F(y,t)$. Since F is continuous, $F^{-1}(U_{F(y,t)})$ is open, so we can find $y \in N_y \in \mathcal{T}_Y$ & $t\in I_t \subseteq I$ s.t. $(y,t) \in N_y \times I_t \subseteq F^{-1}(U_{F(y,t)})$. By slightly shrinking I_t to some (a_t, b_t), we can also conclude $N_t \times [a_t, b_t] \subseteq F^{-1}(U_{F(y,t)})$. Now I is compact, so finitely many $N_{y_1}\times I_{t_1}, \ldots, N_{y_n}\times I_{t_n}$ cover $\{y\}\times I$. Hence $N = N_1 \cap \ldots \cap N_n$ has $N\times I_{t_1}, \ldots, N\times I_{t_n}$ covers $\{y\}\times I$. Putting this all together: we can find $0 = t_1 < \ldots < t_{n+1} = 1$ s.t. $F(N\times[t_i, t_{i+1}])$ lies entirely in an evenly covered open set (call it U_i). We build our lift in n steps by induction:

On $N\times[t_1,t_2]$: We are given the lift on $N\times\{0\}$: $\tilde{F}|_{N\times\{0\}}$. $\tilde{F}(y,0) \in p^{-1}(U_1) = \coprod_{i\in I_1} U_{1,i}$, so $\exists\, i_1$ s.t. $\tilde{F}(y,0) \in U_{1,i_1}$. By possibly shrinking N (replacing it with $N \cap \tilde{F}^{-1}(U_{1,i})$), we may assume $\tilde{F}(n,0) \in U_{1,i_1}$ $\forall n \in N$. Now let $p_{1,i_1} = p|_{U_{1,i_1}}: U_{1,i_1} \xrightarrow{\cong} U_1$, and let $\tilde{F}|_{N\times[t_1,t_2]}: N\times[t_1,t_2] \to \tilde{X}$ be $p_{1,i_1}^{-1} \circ F|_{N\times[t_1,t_2]}$.

Assume built on $N\times[t_1,t_m]$. $\Rightarrow$ A continuous map $\tilde{F}: N\times\{t_m\} \to \tilde{X}$ lifting F. So repeat the argument, extending $\tilde{F}|_{N\times\{t_m\}}$ to $N\times[t_m, t_{m+1}]$ that agrees with $\tilde{F}|_{N\times[t_1,t_m]}$ on $N\times\{t_m\}$.
$\Rightarrow$ these glue to give a continuous map $N\times[t_1, t_{m+1}]$ lifting F.

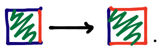
This gives us a continuous function $N\times I \to \tilde{X}$ lifting $\tilde{F}$. A priori, this depends on our starting point $y\in N$, so let's call this $\tilde{F}_y: N_y\times I \to \tilde{X}$. Now if $x \in N_y \cap N_z$, then $\tilde{F}_y|_{\{x\}\times I}$ & $\tilde{F}_z|_{\{x\}\times I}$ are two lifts of $F|_{\{x\}\times I}$. These agree on $(x,0)$, so since I is connected, $\tilde{F}_y|_{\{x\}\times I} = \tilde{F}_z|_{\{x\}\times I}$. So all of our functions $\tilde{F}_y$ glue to give a function $\tilde{F}: Y\times I \to \tilde{X}$, lifting F & extending $\tilde{F}|_{Y\times\{0\}}$. □

Cor (Path lifting) If $\gamma: I \to X$ is a path and $p: \tilde{X} \to X$ is a covering map, then there is a 1-1 correspondence $\{\text{lifts } \tilde{\gamma} \text{ of } \gamma\} \longleftrightarrow p^{-1}(\gamma(0))$.

Pf: Apply the theorem to $Y=\{pt\}$.

Prop Let $\gamma \simeq_{\{0,1\}} \gamma'$ and $p: \tilde{X} \to X$ be a covering map. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be lifts of γ & γ'. If $\tilde{\gamma}(0)=\tilde{\gamma}'(0)$, then $\tilde{\gamma}(1)=\tilde{\gamma}'(1)$ & $\tilde{\gamma} \simeq_{\{0,1\}} \tilde{\gamma}'$.

Pf: There is a homeomorphism $I\times I \to I\times I$ as indicated in the picture:



This means that we can apply the previous theorem to give a lift of a map $I\times I \to X$ with starting data $\tilde{F}(0,t)$, $\tilde{F}(s,0)$, and $\tilde{F}(s,1)$. So let $F: I\times I \to X$ be a htpy rel endpoints $\gamma \simeq_{\{0,1\}} \gamma'$. So $F(s,0)=\gamma(s)$, $F(s,1)=\gamma'(s)$, $F(0,t)=\gamma(0)$, $F(1,t)=\gamma(1)$.

Let $\tilde{x}=\tilde{\gamma}(0)=\tilde{\gamma}'(0)$. Then $c_{\tilde{x}}$ is a lift of $c_{\gamma(0)}$, and hence the unique such map.

Let $\tilde{F}: \{0\}\times I \cup I\times\{0\} \cup I\times\{1\} \longrightarrow \tilde{X}$ be defined by $\tilde{F}(0,t)=\tilde{x}$, $\tilde{F}(s,0)=\tilde{\gamma}(s)$, $\tilde{F}(s,1)=\tilde{\gamma}'(s)$.

These agree on overlaps & are continuous $\Rightarrow$ $\tilde{F}$ is continuous. By the theorem, we have $\tilde{F}$, an ext'n of $\tilde{F}$ over $I\times I$ that lifts F. $\tilde{F}(0,t)=\tilde{x}$, $\tilde{F}(s,0)=\tilde{\gamma}(s)$, $\tilde{F}(s,1)=\tilde{\gamma}(s)$. Now $\tilde{F}(1,t)$ is a lift of the constant path at $\gamma(1)$, starting at $\tilde{\gamma}(1)$. $\Rightarrow$ $\tilde{F}(1,t)$ is constant @ $\tilde{\gamma}(1)$. $\Rightarrow \tilde{\gamma}(1)=\tilde{\gamma}'(1)$, and $\tilde{F}$ is a htpy rel endpoints $\tilde{\gamma}\simeq\tilde{\gamma}'$. □

This is all we need to do some computations!

Thm $\pi_1(S^1,1)\cong\mathbb{Z}$, generated by $t\mapsto e^{2\pi i t}$.

Prop The map $\mathbb{R}\xrightarrow{exp} S^1$ given be $exp(t)=e^{2\pi i t}$ is a covering map.

Pf: Let $z\in S^1$, and let t_0 be any element of $\mathbb{R}$ with $exp(t_0)=-z$. Then $exp^{-1}(S^1-\{-z\})$

$= \mathbb{R}-\{t_0+n \mid n\in\mathbb{Z}\}$ (sin & cos are 2π-periodic but not less than that).

$= \ldots \cup (t_0+n, t_0+n+1) \cup (t_0+n+1, t_0+n+2) \cup \ldots$ and each is $\cong$ to $S^1-\{-z\}$. □

Pf of Thm: Given $[\gamma]\in\pi_1(S^1,1)$, let $\tilde{\gamma}$ be a lift starting at zero. Then $\tilde{\gamma}(1)\in exp^{-1}(1)=\mathbb{Z}$. Any two maps $I\to\mathbb{R}$ are homotopic rel endpoints, so $\tilde{\gamma}\simeq_{\{0,1\}} \underbrace{(t\mapsto n\cdot t)}_{\tilde{\gamma}_n}$, where $n=\tilde{\gamma}(1)$. Note that a homotopy rel endpoints $\tilde{\gamma}\simeq_{\{0,1\}}\tilde{\gamma}_n$ gives one $\gamma\simeq_{\{0,1\}} exp\circ\tilde{\gamma}_n$. So we have a $\mathbb{Z}$s worth of maps: $\gamma_n(t)=e^{2\pi i n t}$. However, direct computation shows $[\gamma_1]^n=[\gamma_n]$, and done! □