It is <u>extremely</u> hard to compute π_1 of a space (hence a lack of examples!). Here is what we can compute for now.

Prop If Y is a convex set in $\mathbb{R}^n$, then $\forall y \in Y$, $\pi_1(Y,y) = \{0\}$.

Pf: Let $\alpha, \beta : I \to Y$ with $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$. Then let $F : I \times I \to Y$ be $F(s,t) = t \cdot \alpha(s) + (1-t)\beta(s)$. Since $\alpha(0) = \beta(0)$, $F(0,t) = t\cdot\alpha(0) + (1-t)\cdot\alpha(0) = \alpha(0)$, and similarly for $F(1,t)$. Now since mult & add in $\mathbb{R}^n$ are continuous, F is continuous. This is a homotopy rel endpoints. Applying this to α a loop & $\beta = c_{\alpha(0)}$ gives the result. □

For other spaces, we work more geometrically (and follow Poincaré's approach).

Def A continuous map $p : Y \to X$ is a <u>covering map</u> if $\forall x \in X$, $\exists\, U \ni x$, open s.t. $p^{-1}(U) = \bigsqcup U_i$, $p|_{U_i} : U_i \to U$ a homeomorphism.

"disjoint union" i.e. $p^{-1}(U) = \bigcup_{i \in I} U_i$ & if $i \neq j$, then $U_i \cap U_j = \emptyset$.

This is a huge, fascinating constraint. We will prove the following incredible theorem.

Thm If X is path connected, $x \in X$, there is a natural bijection

$$\left\{ \begin{array}{c}\text{path-conn.}\\ \text{Covering spaces}\end{array} \tilde{X} \xrightarrow{p} X \ + \ \tilde{x} \in p^{-1}(x) \right\} / \cong \ \xrightarrow{\cong} \ \{\text{subgroups of } \pi_1(X,x)\}.$$

$$(\tilde{X}, \tilde{x}) \xrightarrow{p} (X,x) \longmapsto \operatorname{Im}\left(p_* : \pi_1(\tilde{X},\tilde{x}) \to \pi_1(X,x)\right).$$

There is a lot to unpack here. The map was what we had Monday.

Def: Let $p : E \to X$ & $f : Y \to X$ be continuous functions. A <u>lift</u> of f over p is a map $\tilde{f} : Y \to E$ s.t. $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Thm (Unique lifting) Let Y be connected, let $f : Y \to X$, let $p : \tilde{X} \to X$ be a covering map. Let $g, h : Y \to \tilde{X}$ be two lifts of f. If $\exists\, y \in Y$ s.t. $g(y) = h(y)$, then $\forall y \in Y$, $g(y) = h(y)$.

Pf: Let $E = \{y \in Y \mid g(y) = h(y)\}$ and $F = \{y \in Y \mid g(y) \neq h(y)\}$. Then $E \cap F = \emptyset$ & $E \cup F = Y$. We will show both are clopen. For each $x \in X$, let U_x be such that $p^{-1}(U_x) = \bigsqcup_{i \in I} U_i$ with $p_i = p|_{U_i} : U_i \to U_x$ a homeomorphism. Then for each $y \in f^{-1}(U)$, $\exists\, i$ & j s.t.

$g = p_i^{-1} \circ f$ & $h = p_j^{-1} \circ f$. So if $i \neq j$, then $\forall y \in f^{-1}(U)$, $g(y) \in U_i$, $h(y) \in U_j \Rightarrow$

$g(y) \neq h(y)$. $\Rightarrow$ $f^{-1}(U_x) \subseteq F$. If $i = j$, then $g = h$ on $f^{-1}(U_x)$, and $f^{-1}(U_x) \subseteq E$. Since $\forall y \in Y$, $y \in f^{-1}(U_{f(y)})$, we see both F and E are open. Y connected & $E \neq \emptyset \Rightarrow Y = E$. $\square$