Last time, we finished with the definition of the fundamental group:

$$\pi_1(X,x_0) = \{[\alpha] \mid \alpha : I \to X,\ \alpha(0)=\alpha(1)=x_0\} \quad \text{w/} \quad [\beta]\cdot[\alpha] = [\beta * \alpha]$$

Today we will address 2 points:

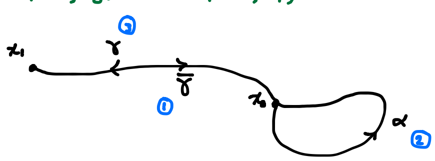
1. How does this depend on x_0?
2. If $f: X \to Y$ is continuous, then what happens on π_1?

I. Change-of-basepoint

Let x_0 & x_1 be points in the same path component & choose a path γ from x_0 to x_1.

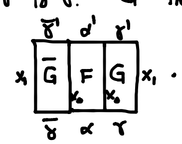
We can use this to compare $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$:

<u>Def</u> Let $h_\gamma : \pi_1(X,x_0) \longrightarrow \pi_1(X,x_1)$ be $h_\gamma([\alpha]) = [\gamma * \alpha * \bar{\gamma}]$, where $\bar{\gamma}(t) = \gamma(1-t)$.



<u>Prop</u>: h_γ is well-defined & If $\gamma \simeq_{\{0,1\}} \gamma'$, then $h_\gamma = h_{\gamma'}$.

<u>Pf</u>: Let F be a homotopy rel $\{0,1\}$ from α to α' and let G be a homotopy rel $\{0,1\}$ from γ to γ'. G then gives $\bar{G}$, a htpy rel $\{0,1\}$ from $\bar{\gamma}$ to $\bar{\gamma}'$. These glue as



$\Rightarrow \gamma * \alpha * \bar{\gamma} \simeq_{\{0,1\}} \gamma' * \alpha' * \bar{\gamma}'$. Letting $\gamma = \gamma'$, G the constant homotopy gives the first part & $\alpha = \alpha'$ the second. $\square$

<u>Prop</u>: If γ_0 is a path $x_0 \to x_1$ & γ_1 is a path $x_1 \to x_2$, then $h_{\gamma_1 * \gamma_0} = h_{\gamma_1} \circ h_{\gamma_0}$.

<u>Pf</u>: $h_{\gamma_1} \circ h_{\gamma_0}([\alpha]) = h_{\gamma_1}([\gamma_0 * \alpha * \bar{\gamma}_0]) = [\gamma_1 * (\gamma_0 * \alpha * \bar{\gamma}_0) * \bar{\gamma}_1] = [(\gamma_1 * \gamma_0) * \alpha * (\bar{\gamma}_0 * \bar{\gamma}_1)] = (1)$. Now,

$\bar{\gamma}_0 * \bar{\gamma}_1 = \overline{(\gamma_1 * \gamma_0)}$, so $(1) = [(\gamma_1 * \gamma_0) * \alpha * \overline{(\gamma_1 * \gamma_0)}] = h_{\gamma_1 * \gamma_0}([\alpha])$. $\square$

<u>Cor</u>: $h_{\bar{\gamma}} = h_\gamma^{-1}$.

<u>Pf</u>: $h_{\bar{\gamma}} \circ h_\gamma = h_{\bar{\gamma} * \gamma} = h_{c_{x_0}}$. Now $h_{c_{x_0}}([\alpha]) = [\bar{c}_{x_0} * \alpha * c_{x_0}] = [\alpha]$. The other direction is identical. $\square$

<u>Cor</u>: h_γ is bijective for all γ.

This also respects the product.

<u>Prop</u> h_γ is a homomorphism.

Pf We compute: $h_\gamma([\alpha]\cdot[\beta]) = h_\gamma([\alpha*\beta]) = [\gamma*\alpha*\beta*\bar{\gamma}] = [\gamma*\alpha*c_{x_0}*\beta*\bar{\gamma}] = [\gamma*\alpha*(\bar{\gamma}*\gamma)*\beta*\bar{\gamma}]$

$= [(\gamma*\alpha*\bar{\gamma})*(\gamma*\beta*\bar{\gamma})] = [\gamma*\alpha*\bar{\gamma}]\cdot[\gamma*\beta*\bar{\gamma}] = h_\gamma([\alpha])\cdot h_\gamma([\beta])$. $\square$

Putting this all together gives the dependence on the base point.

Thm If x_0 & $x_1 \in X$ are in the same path componet, then $\pi_1(X,x_0) \cong \pi_1(X,x_1)$. Any path γ from x_0 to x_1 gives us an isomorphism h_γ.

Remark ① If $x_0 = x_1$, then we get an automorphism of $\pi_1(X,x_0)$: $h_\alpha(\beta) = \alpha\beta\alpha^{-1}$. This is an important construction in group theory: conjugation.

② If x and y are in different path components, we can't say anything about $\pi_1(X,x)$ & $\pi_1(X,y)$.

II. Functions & π_1.

If $f: X \to Y$ & if $\alpha: I \to X$ is a path, then $f\circ\alpha: I \to Y$ is a path in Y. We can use this to produce a map of fundamental groups.

Thm If $f: X \to Y$ is continuous & $y = f(x)$, then the function $f_*: \pi_1(X,x) \longrightarrow \pi_1(Y,y)$ defined by $f_*([\alpha]) = [f\circ\alpha]$ is a homomorphism. Moreover, if $f \underset{\{x\}}{\simeq} g$, then $f_* = g_*$.

Pf: We saw last time that if $f \underset{\{x\}}{\simeq} g$ & $\alpha \underset{\{0,1\}}{\simeq} \alpha'$, then $f\circ\alpha \underset{\{0,1\}}{\simeq} g\circ\alpha'$. This shows both well-definedness of f_* & that $f_* = g_*$.

To see f_* is a homomorphism, we check:

$f_*[\alpha*\beta] = [f\circ(\alpha*\beta)]$. Now $f\circ(\alpha*\beta)(t) = \begin{cases} f\circ\alpha(2t) & 0 \le t \le \frac{1}{2} \\ f\circ\beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} = (f\circ\alpha)*(f\circ\beta)$, so

$= f_*(\alpha)\cdot f_*(\beta)$, as desired.

Remark: Combining the change-of-base point with this theorem explains how to compare f_* & g_* when they are homotopic but not so rel x. Exercise!