Def A <u>path</u> <u>in X</u> <u>based</u> <u>at</u> <u>x</u> is a continuous $\alpha : I \longrightarrow X$ with $\alpha(0)=x$.

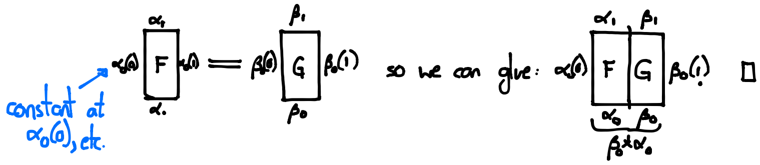
Just as before, we can glue paths.

Def If α & β are paths in X with $\alpha(1)=\beta(0)$, then let $\beta * \alpha = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$

This isn't well-behaved as is, but it is up to homotopy.

Prop: If $\alpha_0 \simeq_{\{0,1\}} \alpha_1$ & $\beta_0 \simeq_{\{0,1\}} \beta_1$, then $\beta_0 * \alpha_0 \simeq_{\{0,1\}} \beta_1 * \alpha_1$. i.e. $*$ descends to htpy classes: $\boxed{[\beta]*[\alpha]=[\beta*\alpha]}$

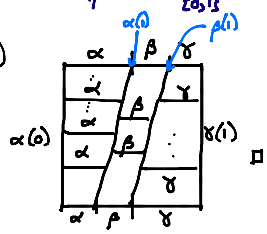
Pf (by picture) Let F be a htpy rel $\{0,1\}$ $\alpha_0 \simeq \alpha_1$ & G one for βs. We can depict this as



$\square$

Prop We have $\gamma * (\beta * \alpha) \simeq_{\{0,1\}} (\gamma * \beta) * \alpha$. i.e. $[\gamma]*([\beta]*[\alpha]) = ([\gamma]*[\beta])*[\alpha]$

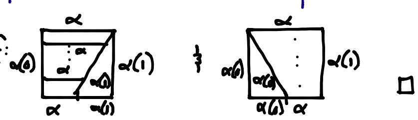
Pf: (by picture)



$\square$

Prop: If c_x is the constant path at x, then $c_{\alpha(1)} * \alpha \simeq_{\{0,1\}} \alpha$, $\alpha * c_{\alpha(0)} \simeq_{\{0,1\}} \alpha$

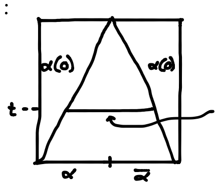
Pf:



$\square$

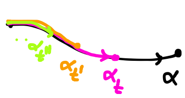
Prop If $\bar{\alpha}(t)=\alpha(1-t)$, then $\alpha * \bar{\alpha} \simeq_{\{0,1\}} c_{\alpha(1)}$ & $\bar{\alpha} * \alpha \simeq_{\{0,1\}} c_{\alpha(0)}$.

Pf:



$\bar{\alpha}_t * \alpha_t$, where $\alpha_t(s) = \alpha(s\cdot(1-t))$ traces α from 0 to $1-t$. $\square$

This one is the trickiest, so here is the idea. A time t, we run α from 0 to $1-t$, then we immediately turn around. As t varies, this goes between $\bar{\alpha} * \alpha$ and then not doing anything!



Def A <u>loop</u> in X based at x is a path $\gamma : I \longrightarrow X$ s.t. $\gamma(0)=\gamma(1)=x$.

Def The <u>fundamental</u> <u>group</u> of X based at x is $\pi_1(X,x) = \{[\gamma] \mid \gamma \text{ is a loop based at } x\}$.

Prop $\pi_1(X,x)$ is a group under $*$: $[\gamma]*[\delta]=[\gamma*\delta]$.

Pf $*$ is associative on htpy classes, and c_x is the identity. If $[\gamma]\in\pi_1(X,x)$, then $[\bar{\gamma}]$ has $[\gamma]*[\bar{\gamma}]=[c_x]=[\bar{\gamma}]*[\gamma]$. $\square$

<u>Remark</u>: We are only remembering a little bit of structure. What we have really shown is that we have a category $\tilde{X}$ with objects: $\{x\in X\}$ and $\mathrm{Hom}(x,y)=\{[\gamma]\mid \gamma \text{ a path } \gamma(0)=x, \gamma(1)=y\}$. Then $*$ gives us a composition. This is the "fundamental groupoid of X."