Def A group is a set G + a multiplication $\begin{array}{c} G\times G \longrightarrow G \\ (g,h) \longmapsto g\cdot h \end{array}$ s.t.

1) $g\cdot(h\cdot k) = (g\cdot h)\cdot k$ (mult. is associative)

2) $\exists e \in G$ s.t. $g\cdot e = e\cdot g = g \quad \forall g \in G$ (there is a mult. identity)

3) $\forall g \in G$, $\exists g' \in G$ s.t. $g\cdot g' = g'\cdot g = e$. (there are mult. inverses)

Ex: 1) $(\mathbb{Z},+)$ is a group.

2) $(\mathbb{R},+)$ is a group

3) Any vector space has an underlying additive group (vector addition)

4) Consider $\{0,\dots,n-1\}$ and let $a+b =$ the remainder when we compute $(a+b)/n$.

For the last one, we can record the mult with a table.

$\mathbb{Z}/3$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

0 is the identity

every row just rearranges the top

same with the columns.

Prop: e is unique.

Pf: If e & e' satisfy $e\cdot g = g\cdot e = g = g\cdot e' = e'\cdot g \quad \forall g \in G$, then

$$e = e\cdot e' = e' \qquad \square$$

Prop Inverses are unique

Pf: If g' & g'' both satisfy $g\cdot g' = g'\cdot g = e = g\cdot g'' = g''\cdot g$, then

$$g' = g'\cdot e = g'\cdot(g\cdot g'') = (g'\cdot g)\cdot g'' = e\cdot g'' = g'' \qquad \square$$

Def A subset $\emptyset \neq H \subseteq G$ is a subgroup if $\forall h_1,h_2 \in H$, $h_1\cdot h_2 \in H$ & $h_1^{-1} \in H$.

Prop A subgroup is a group under $\cdot$ & H is a s.g. iff $\forall h_1,h_2 \in H$, $h_1\cdot h_2^{-1} \in H$.

Pf: ① Since $\cdot$ is associative for G, it is here too. If $h \in H$, then $h^{-1} \in H$ & $e = h\cdot h^{-1} \in H$. So there is an identity & there are inverses. That this satisfies the axioms follows from G.

② $\Rightarrow$) If $h_1,h_2 \in H$, then $h_2^{-1} \in H \Rightarrow h_1\cdot h_2^{-1} \in H$.

$\Leftarrow$) If $h \in H$, then $h\cdot h^{-1} = e \in H$. $\Rightarrow$ $e\cdot h^{-1} = h^{-1} \in H$. Now if $h_1,h_2 \in H$, then $h_1\cdot h_2 = h_1\cdot (h_2^{-1})^{-1} \in H$. $\square$

Inverses are unique

Def A map $f: G \to H$ is a <u>homomorphism</u> if $\forall g_1, g_2 \in G$, $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$.

Prop 1) $f(e_G) = e_H$

2) $f(g^{-1}) = f(g)^{-1}$.

We have a lemma.

Lem If $g \in G$ satisfies $g \cdot g = g$, then $g = e_G$.

Pf: $g^{-1}(g \cdot g = g)$ gives
$$g^{-1}(g \cdot g) = g^{-1} \cdot g = e$$
$$(g^{-1} g) \cdot g = e \cdot g = g \quad \square$$

Pf of Prop 1) $e_G \cdot e_G = e_G$ so $f(e_G) \cdot f(e_G) = f(e_G \cdot e_G) = f(e_G)$. $\Rightarrow f(e_G) = e_H$.

2) $f(g) \cdot f(g^{-1}) = f(g \cdot g^{-1}) = f(e_G) = e_H$. $\Rightarrow f(g^{-1}) = f(g)^{-1}$. $\square$

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Def Let $f, g: X \to Y$. A <u>homotopy</u> $f \simeq g$ is a continuous map $F: X \times I \to Y$ s.t. $F(x,0) = f(x)$ & $F(x,1) = g(x)$. We say f & g are <u>homotopic</u>.

Prop $\simeq$ is an equivalence relation on $\mathrm{Map}(X, Y)$.

Pf: $f \simeq f$: Let $F: X \times I \xrightarrow{\pi_X} X \xrightarrow{f} Y$

$f \simeq g \Rightarrow g \simeq f$: If F is a homotopy, then let $\bar{F}: X \times I \xrightarrow{\mathrm{Id} \times \{t \mapsto 1-t\}} X \times I \xrightarrow{F} X$.

$f \simeq g, g \simeq h$: If $f \overset{F}{\simeq} g$ & $g \overset{G}{\simeq} h$, then let $H: X \times I \xrightarrow{\cong} X \times [0,2] \xrightarrow{F \cup G} Y$. $\square$
$$(x,t) \longmapsto (x, 2t)$$

Prop If $f_0 \simeq g_0$ and $f_1 \simeq g_1$, then $f_1 \circ f_0 \simeq g_1 \circ g_0$.

Pf Let F_i be a homotopy $f_i \simeq g_i$. Define F by
$$X \times I \longrightarrow Y \times I \xrightarrow{F_1} Z$$
$$(x,t) \longmapsto (F_0(x,t), t) \mapsto F_1(F_0(x,t), t).$$
The first map is continuous since each projection is.

Then $F(x,0) = F_1(F_0(x,0),0) = F_1(f_0(x),0) = f_1(f_0(x))$ & $F(x,1) = F_1(F_0(x,1),1) = F_1(g_0(x),1) = g_1(g_0(x))$. $\square$

Cor If $f \simeq g$, then $h \circ f \simeq h \circ g$ & $f \circ h \simeq g \circ h$.

Def If $A \subseteq X$ is closed & $f, g: X \to Y$, and if $\forall a \in A$, $f(a) = g(a)$, then we say f <u>is homotopic to</u> g <u>relative to</u> A if $\exists$ a homotopy $F: X \times I \to Y$ s.t. $\forall a \in A$, $F(a,t) = f(a) = g(a)$.

The obvious relative version of the previous prop holds.