MATH 121 – MIDTERM II

NAME:

Problem 1. *Give the definitions of the following terms:*

(1) *Hausdorff, Regular, & Normal*

Hausdorff: given $x \neq y \in X$, $\exists\, U \ni x, V \ni y$, $U \cap V = \emptyset$.

Regular: given $x \in X$, $C \subseteq X$ closed, $x \notin C$, $\exists\, U \ni x, V \supseteq C$ s.t. $U \cap V = \emptyset$.

Normal: given C_1, C_2 closed, $C_1 \cap C_2 = \emptyset$, $\exists\, U \supseteq C_1, V \supseteq C_2$ s.t. $U \cap V = \emptyset$.

(2) *Connected*

X is connected if the only clopen sets are X & $\emptyset$.

(3) *Basis for a topology*

$\mathcal{B} \subseteq \tau_X$ is a basis if $\forall\, U \in \tau_X$, U is a union of elements of $\mathcal{B}$.

(4) *Continuous*

$f : X \longrightarrow Y$ is continuous if $\forall\, U \in \tau_Y$, $f^{-1}(U) \in \tau_X$.

Problem 2. *Let X be a space and let $A \subset X$ be a closed subspace. Let $f\colon A \to \mathbb{R}^n$ be a continuous function*

$$f(x) = \big(f_1(x), \ldots, f_n(x)\big)$$

such that for $1 \le i \le n$, f_i is a bounded function. Show that there is a continuous function $F\colon X \to \mathbb{R}^n$ such that $F(a) = f(a)$ for all $a \in A$.

By the Tietze extension theorem, for each i, there is a map $F_i : X \longrightarrow \mathbb{R}$ extending f_i. Let $F : X \longrightarrow \mathbb{R}^n$ be $F(x) = (F_1(x), \ldots, F_n(x))$. The composite of F with the i^{th} projection is F_i, which is continuous, so F is continuous. Since each F_i extends f_i, F extends f.

Problem 3. *Let X be a space and let Y and Z be closed subspaces. Let $f\colon Y \to W$ and $g\colon Z \to W$ be functions such that for all $x \in Y \cap Z$ $f(x) = g(x)$. Show that*

$$f \cup g\colon Y \cup Z \to W$$

is continuous if and only if f and g are.

Let $C \subseteq W$ be closed.

$\Leftarrow$ f, g continuous $\Rightarrow$ $f^{-1}(C)$ closed in Y, $g^{-1}(C)$ closed in Z. Since Y & Z are closed in X, $f^{-1}(C)$ & $g^{-1}(C)$ are closed in X & hence $f^{-1}(C) \cup g^{-1}(C)$ is closed in $X \Rightarrow$ in $Y \cup Z$. Now $f^{-1}(C) \cup g^{-1}(C) = (f \cup g)^{-1}(C)$.

$\Rightarrow$ $Y \hookrightarrow Y \cup Z$ is continuous, so the composite $Y \longrightarrow Y \cup Z \longrightarrow W$ is. This is just f. The same result gives g's continuity.

Problem 4. *Let X and Y be spaces.*

(1) *Show that if Y is compact, then the projection $\pi_X : X \times Y \to X$ is closed (i.e. takes closed sets to closed sets).*

Let $V \subseteq X \times Y$ be closed. If $\pi_X(V) = X$, then we are done, since X is closed. So assume $x \notin \pi_X(V) \Leftrightarrow \{x\} \times Y \cap V = \emptyset$. Since V is closed, $\forall y \in Y$, there are opens U_y & U_y' s.t. $(x,y) \in U_y \times U_y' \subseteq X \times Y - V$. Since Y is compact, finitely many U_y' cover: $U_{y_1}', \dots, U_{y_n}'$. Let $U = U_{y_1} \cap \dots \cap U_{y_n}$. Then $\{x\} \times Y \subseteq (U \times U_{y_1}' \cup \dots \cup U \times U_{y_n}') = U \times Y$, and $U \times Y \cap V = \emptyset$ (since $U \times U_{y_i}' \subseteq U_{y_i} \times U_{y_i}' \subseteq X \times Y - V$). $\Rightarrow U \cap \pi_X(V) = \emptyset$, and any $x \in X - \pi_X(V)$ has an open neighborhood in $X - \pi_X V$.

(2) *Show that if Y is compact and Hausdorff, then a function $f : X \to Y$ is continuous if and only if the graph*

$$\Gamma_f = \{(x, f(x)) | x \in X\} \subset X \times Y$$

is closed.

Y is Hausdorff iff $\Delta_Y = \{(y,y) | y \in Y\}$ is closed.

$\Rightarrow$) Consider $X \times Y \xrightarrow{f \times 1} Y \times Y$ defined by $(x,y) \mapsto (f(x), y)$. If f is continuous, then so is this: Let π_Y^1 & π_Y^2 be the two projections $Y \times Y \to Y$. Then $\pi_Y^2 \circ (f \times 1)(x,y) = y = \pi_Y(x,y)$, so $\pi_Y^2 \circ (f \times 1)$ is continuous. $\pi_Y^1 \circ (f \times 1)(x,y) = f(x)$, so $\pi_Y^1 \circ (f \times 1) = f \circ \pi_X$ is continuous.

$(f \times 1)^{-1}(\Delta) = \Gamma_f$ is then closed.

$\Leftarrow$) Let $C \subseteq Y$ be closed. Then $\pi_Y^{-1}(C)$ is closed $\Rightarrow \pi_Y^{-1}(C) \cap \Gamma_f$ is closed.

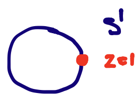
Now $\pi_Y^{-1}(C) \cap \Gamma_f = \left\{(x,y) \,\middle|\, \begin{matrix} f(x) = y \\ y \in C \end{matrix}\right\} = \left\{(x,y) \,\middle|\, \begin{matrix} x \in f^{-1}(C) \\ y = f(x) \end{matrix}\right\}$.

Since Y is compact, π_X is closed $\Rightarrow \pi_X(\pi_Y^{-1}(C) \cap \Gamma_f) = f^{-1}(C)$ is closed.

Problem 5. *If X is a discrete space, then show that there is a minimal set of open sets $\mathcal{B}$ that is contained in every basis for the topology on X. What is this set?*

X is discrete iff $\forall x \in X$, $\{x\}$ is open. If $\mathcal{B}'$ is a basis, then $\forall x \in X$, $\{x\}$ is a union of basic open sets $\Rightarrow$ $\{x\}$ is a basic open set. So if $\mathcal{B}'$ is a basis, then $\{\{x\} \mid x \in X\} \subseteq \mathcal{B}'$. This is a basis, since every set is a union of points, and if we were missing any $y \in X$, then $\{y\}$ would be missed.

Problem 6. *Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $A \subset S^1$ be $\{z \in S^1 \mid z \neq 1\}$. Define an equivalence relation on S^1 by $x \sim y$ if $x = y$ or if x and y are both in A. Describe explicitly the space $S^1/\sim$. How many points does it have? Is it compact? Is it connected? Is it T_0? T_1? T_2? T_3? T_4?*



$\Rightarrow$ $S^1/\sim \; = \{[1], \underbrace{[-1]}_{= [z] \mid z \neq 1}\}$. $U \subseteq S^1/\sim$ is open iff $q^{-1}(U)$ is open.

$q^{-1}(\{[1]\}) = \{1\}$: not open

$q^{-1}(\{[-1]\}) = S^1 - \{1\}$: open

$\left.\begin{array}{l} q^{-1}(\emptyset) = \emptyset \\ q^{-1}(S^1/\sim) = S^1 \end{array}\right\}$ open.

So this is that interesting 2 point space.

Compact? Yes! It's finite

Connected? Yes! $\{1\}$ is closed but not open.

T_0? Yes! (Saw in class)

T_1? No! $[-1]$ is not closed.

$T_{>1}$? No! $T_{>1} \Rightarrow T_1$.