We have seen that compact Hausdorff spaces are much nicer than ordinary spaces. We'd like to embed an arbitrary space in a compact one. This is called a "compactification."

Def Let X be a Hausdorff space, and let $X^+ = X \cup \{\infty\}$. Define a collection of subsets on X^+ by $\tau_{X^+} = \tau_X \cup \{X^+ - K \mid K \subseteq X \text{ is compact}\}$. X^+ is the <u>1-point compactification</u>.

Prop τ_{X^+} is a topology on X^+.

Pf: ① $\emptyset \in \tau_X$ & $\emptyset$ compact $\Rightarrow X^+ \in \tau_{X^+}$.

② Let $\{U_i\}_{i\in I} \subseteq \tau_{X^+}$. If all U_i are in τ_X, then by assumption $\bigcup_{i\in I} U_i \in \tau_X \subseteq \tau_{X^+}$.

If for some i, $U_i = X^+ - K$, K compact, then $\bigcup_{i\in I} U_i = (X^+ - K) \cup \bigcup_{j \in I - \{i\}} U_j = X^+ - \left(K \cap \bigcap_{j\in I-\{i\}} X^+ - U_j\right) - (*)$

Now for all $U \in \tau_{X^+}$, $X \cap (X^+ - U)$ is closed: if $U \in \tau_X$, then $X \cap (X^+ - U) = X - U$ is closed. If $U = X^+ - K$, then $X \cap (X^+ - K) = K$, a compact in X. Since X is Hausdorff, K is closed.

So $(*) = X^+ - \left(K \cap \bigcap_j X^+ - U_j\right) \overset{\text{since } K \subseteq X}{=} X^+ - \left(K \cap \bigcap X \cap (X^+ - U_j)\right) = X^+ - (K \cap V)$, V a closed in X. $\Rightarrow K \cap V$, being closed in a compact, is compact.

③ Let $U_1, \dots, U_n \in \tau_{X^+}$. If at least one of the $U_i \in \tau_X$ (say, U_1), then

$U_1 \cap U_2 \cap \dots \cap U_n = U_1 \cap (X \cap U_2) \cap \dots \cap (X \cap U_n)$. $\forall U \in \tau_{X^+}$, $U \cap X \in \tau_X$ (by the above closed argument),

so this is in τ_X. If all U_i are of the form $U_i = X^+ - K_i$, then

$U_1 \cap \dots \cap U_n = X^+ - (K_1 \cup \dots \cup K_n)$. Finite unions of compacts are compact, so this is in τ_{X^+}. □

Prop X^+ is compact.

Pf Let $\{U_i\}_{i\in I}$ be an open cover of X^+. For some $i_0 \in I$, $\infty \in U_{i_0}$, so $U_{i_0} = X^+ - K$. Now $\{U_i \cap X\}_{i \in I - \{i_0\}}$ is an open cover of the compact K, so we have a finite subcover $\{U_{i_1} \cap X, \dots, U_{i_n} \cap X\}$.

Then $\{U_{i_0}, \dots, U_{i_n}\}$ is a cover of X^+ □

To get Hausdorff, we need that points in X can be separated from ∞: $\exists\, x \in U$, $\infty \in V$, $V \cap U = \emptyset$. So can assume $U \subseteq K$, where $V = X^+ - K$.

Def X is <u>locally compact</u> if $\forall x \in X$, $\exists\, U \in \tau_X$, $x \in U$ s.t. $\overline{U}$ is compact.

Prop If X is locally compact and Hausdorff, then X^+ is Hausdorff.

Pf: If $x,y \in X^+$ are both in X, then X Hausdorff gives the needed opens. If $x \in X$ & $y = \infty \in X^+$, then

Let $U \ni x$ be such that $\bar{U}$ is compact. Then $V = X^+ - \bar{U} \ni \infty$ & $U \cap V = \emptyset$. □

The above description shows that have open sets separating x & ∞ iff X is loc. comp.

Thm If $\{X_\alpha\}_{\alpha \in A}$ is a collection of compact spaces, then $\prod_{\alpha \in A} X_\alpha$ is compact.

The proof uses Zorn's Lemma (and the theorem is equivalent to it):

Zorn's Lemma If $\mathcal{P}$ is a partially ordered set s.t. any totally ordered subset has an upper bound, then $\mathcal{P}$ has a maximal element.

So there is a lot to unpack. ① A partially ordered set is a set P + a binary operation $\leq$ s.t. ⓘ If $a \leq b$, $b \leq a \Rightarrow a = b$, ⓘⓘ $a \leq a$, ⓘⓘⓘ $a \leq b$, $b \leq c \Rightarrow a \leq c$. ② A totally ordered set is a partially ordered set s.t. $\forall a,b$, $a \leq b$ or $b \leq a$. ③ An upper bound for $S \subseteq P$ is an element $c \in P$ s.t. $\forall a \in S$, $a \leq c$. ④ A maximal element is an $m \in P$ s.t. $m \leq c \Rightarrow m = c$.

A prototype for the use of this is the following.

Thm Every vector space has a basis.

Pf: Let $\mathcal{P} = (\{\text{linearly independent subsets of } V\}, \subseteq)$. This is a poset. If $\{L_i\}_{i \in I}$ is a totally ordered subset, then let $L = \bigcup_{i \in I} L_i$. If $\{v_1, \dots, v_n\} \subseteq L$, then for each $1 \leq j \leq n$, choose an i_j s.t. $v_j \in L_{i_j}$. Since these are totally ordered, $\exists i$ s.t. $L_{i_1}, \dots, L_{i_n} \subseteq L_i$, and hence $\{v_1, \dots, v_n\} \subseteq L_i$, a linearly independent set $\Rightarrow L \in \mathcal{P}$. By Zorn's Lemma, $\mathcal{P}$ has a maximal element L. If $\text{span}(L) \subseteq V$ were proper, then $L \cup \{v\}$, $v \in V - \text{span}(V)$ would be linearly independent & $L \leq$, a contradiction to maximality. □

Our argument works the same way. We prove a criterion for compactness.

Lemma Let $\mathcal{S}$ be a sub-base for X. If every cover of X by elements of $\mathcal{S}$ has a finite subcover, then X is compact.

Pf: Assume X is not compact. This means we have an open cover $\mathcal{C}$ s.t. no finite collection of elements of $\mathcal{C}$ covers. Let $\mathcal{P}$ be the poset of collections $\mathcal{E}$ of open covers s.t. no finite

collection covers. This is a poset under inclusion & is non-empty ($\mathcal{C} \in \mathcal{P}$). If $\{\mathcal{E}_i\}_{i\in I}$ is a totally ordered subset of $\mathcal{P}$, then let $\mathcal{E} = \bigcup_{i\in I} \mathcal{E}_i$. This covers, since each $\mathcal{E}_i$ does, and if $\{U_1,\dots,U_n\} \subseteq \mathcal{E}$, then by total ordering, $\exists\, i$ s.t. $\{U_1,\dots,U_n\} \subseteq \mathcal{E}_i$. $\Rightarrow$ $\{U_1,\dots,U_n\}$ doesn't cover.
(This is the same argument!)
By Zorn's Lemma, $\exists$ a maximal $\mathcal{M}$. Now if $\mathcal{M}_0 = \mathcal{M} \cap \mathfrak{S}$, then this is a collection of sub-basic open sets s.t. no finite collection covers (since it is also a finite set in $\mathcal{M}$). It suffices to then show that $\mathcal{M}_0$ covers.

This is like the "span" part of the argument.

Let $x \in X$ $\exists\, U \in \mathcal{M}$ s.t. $x \in U$. Since $\mathfrak{S}$ is a subbase, there are $V_1,\dots,V_n \in \mathfrak{S}$ s.t. $x \in V_1 \cap \dots \cap V_n \subseteq U$. If no $V_i \in \mathcal{M}$, then $\{V_i\} \cup \mathcal{M}$ must have a finite subcover (by maximality). Let $W_{i1},\dots,W_{im} \in \mathcal{M}$ be s.t. $\{V_i, W_{i1},\dots,W_{im}\}$ is a cover of X. Then $\{V_1 \cap \dots \cap V_n, W_{11},\dots,W_{1m},\dots,W_{nm}\}$ is a cover of X $\Rightarrow$ $\{U, W_{11},\dots,W_{nm}\} \subseteq \mathcal{M}$ is a finite subcover of $\mathcal{M}$, a contradiction. $\Rightarrow$ for some i, $V_i \in \mathcal{M}_0$ & $x \in V_i$. Hence $\mathcal{M}_0$ covers. $\square$

<u>Pf of Theorem</u> Let $\mathfrak{S} = \{\pi_\alpha^{-1}(V) \mid {\alpha \in A \atop V \in \mathcal{T}_{X_\alpha}}\}$, and let $\mathcal{C}$ be a subset of $\mathfrak{S}$ such that no finite subset covers. For each $\alpha \in A$, consider $\{\pi_\alpha^{-1}(V_i)\}_{i \in I_\alpha} \subseteq \mathcal{C}$, the collection of all elements of $\mathcal{C}$ of the form $\pi_\alpha^{-1}(V)$. Since X_α is compact, if $\{V_i\}_{i\in I_\alpha}$ were a cover, then we would be able to find a finite subset $\{V_{i_1},\dots,V_{i_n}\}$ that covers, and hence $\{\pi_\alpha^{-1}(V_{i_1}),\dots,\pi_\alpha^{-1}(V_{i_n})\}$ would cover. $\Rightarrow$ $\{V_i\}_{i\in I_\alpha}$ does not cover. Choose $x_\alpha \in X - \bigcup_{i\in I_\alpha} V_i$. Then let $f: A \longrightarrow \bigcup_{\alpha\in A} X_\alpha$ be $f(\alpha) = x_\alpha$. This is an element of $\prod_{\alpha\in A} X_\alpha$, and by construction, it is not in any of the elements of $\mathcal{C}$. $\square$