Def A set is <u>clopen</u> if it is open & closed.

X is <u>connected</u> if the only clopen sets are $\emptyset$ & X.

In other words, if $X = E \cup F$, $E \cap F = \emptyset$, E & F open, then E or $F = X$.

Prop If X is connected & $f: X \to Y$ is continuous, then $f(X)$ is connected.

Pf: If $f(X) = E \cup F$, $E \cap F = \emptyset$, both open, then $X = f^{-1}(E) \cup f^{-1}(F)$, both open & disjoint.

$\Rightarrow X = f^{-1}(E)$ or $f^{-1}(F)$. $\Rightarrow f(X) \subseteq E$ or $f(X) \subseteq F$. $\square$

Lemma Any interval in $\mathbb{R}$ is connected.

Pf: Any interval is $\cong$ to $[0,1]$, $[0,1)$, $(0,1]$, or $(0,1)$. All the proofs are the same. If $[0,1] = E \cup F$, $E \cap F = \emptyset$, both open ($\leftrightarrow$ both closed). Assume $1 \notin E$ (otherwise swap) & let $t_0 = \sup_{t \in E} \{t\}$. Then E closed $\Rightarrow t_0 \in E$. $1 \notin E \Rightarrow t_0 < 1$. E open $\Rightarrow \exists\, \epsilon$ s.t. $[t_0, t_0+\epsilon) \subseteq E \Rightarrow t_0$ not the sup. $\square$

Def A <u>path</u> in X is a continuous function $[0,1] \xrightarrow{\alpha} X$.

Cor: Any path has connected image.

Def Let $\sim$ be $x \sim y$ iff $\exists$ path α w/ $\alpha(0) = x$, $\alpha(1) = y$.

Prop $\sim$ is an equivalence relation on X.

Pf: Let $c_x : [0,1] \to X$ be $c_x(t) = x$. This shows $x \sim x$.

If $\alpha : I \to X$, then $t \xmapsto{\alpha'} \alpha(1-t)$ has $\alpha'(0) = \alpha(1)$, $\alpha'(1) = \alpha(0)$. So $x \sim y \Rightarrow y \sim x$.

$x \sim y, y \sim z \leftrightarrow \exists\, \alpha, \beta : I \to X$ s.t. $\alpha(0) = x$, $\alpha(1) = \beta(0) = y$, $\beta(1) = z$. Let γ be defined by

$$[0,1] \xrightarrow{\cong} [0,2] \xrightarrow{\gamma} X, \qquad \gamma(t) = \begin{cases} \alpha(t) & t \in [0,1] \\ \beta(t-1) & t \in [1,2] \end{cases}$$
$$t \longmapsto 2t$$

γ is continuous:

If $V \subseteq X$ is closed, then $\gamma^{-1}(V) = \underset{[0,1]}{\underset{\cap}{\alpha^{-1}(V)}} \cup \underset{[1,2]}{\underset{\cap}{\beta^{-1}(V)}}$. α, β continuous $\Rightarrow \alpha^{-1}(V), \beta^{-1}(V)$ closed in $[0,1]$ & $[1,2]$ respectively $\Rightarrow$

$\alpha^{-1}(V), \beta^{-1}(V)$ closed in $[0,2] \Rightarrow \gamma^{-1}(V)$ is closed. $\square$

Def The <u>path components</u> of X are the equivalence classes for this path $\sim$.

Def X is <u>path connected</u> if there is one path component.

In fact, path components are connected.

Prop Let E_α $\alpha \in I$ be a collection of connected subspaces of X s.t. $\forall \alpha \neq \beta$, $E_\alpha \cap E_\beta \neq \emptyset$.
Then $\bigcup_{\alpha \in I} E_\alpha$ is connected.

Pf Let $Y = \bigcup_{\alpha \in I} E_\alpha$, and let $Y = E \cup F$, $E \cap F = \emptyset$, both clopen. Assume $E \neq \emptyset$ (otherwise swap E and F). Since each E_α is connected and since $(E \cap E_\alpha) \cup (F \cap E_\alpha)$ is a disjoint open cover of E_α, we deduce that if $E \cap E_\alpha \neq \emptyset$, then $E \cap E_\alpha = E_\alpha \Rightarrow E_\alpha \subseteq E$. Let $e \in E$. Since $e \in E_\alpha$ some α, $E_\alpha \cap E \neq \emptyset \Rightarrow E_\alpha \subseteq E$. $\forall \beta$, $E_\beta \cap E_\alpha \neq \emptyset$, so $\underbrace{E_\beta \cap E_\alpha \cap E}_{= (E_\beta \cap E) \cap E_\alpha} = E_\beta \cap E_\alpha \neq \emptyset$
$\Rightarrow E_\beta \cap E \neq \emptyset \Rightarrow E_\beta \subseteq E \Rightarrow Y \subseteq E$. $\square$

Cor The path components of X are connected.

Def The <u>connected component</u> of $x \in X$ is the union of all connected subspaces of X that contain x.

Prop The connected component of x is the largest connected subspace of X containing x.

Pf: Let E_x be the connected component of x. By the previous proposition, E_x is connected. Since E_x was the union of all such subspaces, E_x is maximal. $\square$

Cor Connected components are equal or disjoint.

Pf: Let $x, y \in X$ & let E_x & E_y be their connected components. If $E_x \cap E_y \neq \emptyset$, then $E_x \cup E_y$ is a connected space containing x and y. $\Rightarrow E_x = E_x \cup E_y = E_y$. $\square$