Def $Y \subseteq X$ is compact if for every open cover $\{U_i\}_{i\in I}$ of Y, there is a finite set $\{i_1,\dots,i_n\}$ s.t. $Y \subseteq U_{i_1} \cup \dots \cup U_{i_n}$.

The slogan is that "every cover has a finite subcover."

Ex: 1) Any finite set is compact. (there are only finitely many things to check)

2) Any space with the cofinite topology is compact: any non-empty open misses finitely many points, so we only need finitely more.

Prop Let X be compact & let $f: X \to Y$ be continuous. Then $f(X) \subseteq Y$ is compact.

Pf: Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$ in Y. Then since f is continuous, $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $f^{-1}(f(X))$ and hence of X. Since X is compact, for some $\{i_1,\dots,i_n\} \subseteq I$,

$X \subseteq f^{-1}(U_{i_1}) \cup \dots \cup f^{-1}(U_{i_n}) \Rightarrow f(X) \subseteq f(f^{-1}(U_{i_1})) \cup \dots \cup f(f^{-1}(U_{i_n})) \subseteq U_{i_1} \cup \dots \cup U_{i_n}$. Hence we have a finite subcov.

Cor If X is compact, then so is any quotient.

Prop If X is compact & $V \subseteq X$ is closed, then V is compact.

Pf: Let $\{U_i\}_{i\in I}$ be an open cover of V. Since V is closed, $X-V$ is open, and hence $\{X-V\} \cup \{U_i\}_{i\in I}$ is an open cover of X. Since X is compact, this has a finite subcover $\{X-V\} \cup \{U_{i_1},\dots,U_{i_n}\}$ (without loss of generality, we can assume $X-V$ is in the subcover, since adding it in doesn't change "finite"). Then $\{U_{i_1},\dots,U_{i_n}\}$ is a finite subcover of $\{U_i\}_{i\in I}$, and V is compact. $\square$

We have a converse to this when X is Hausdorff.

Prop If X is Hausdorff, then if $K \subseteq X$ is compact, then K is closed.

Pf: Let $K \subseteq X$ be compact & let $x \in X-K$. Since X is Hausdorff, $\forall k \in K$, $\exists$ disjoint opens $U_k \ni k$ & $V_k \ni x$. Now $K \subseteq \bigcup_{k\in K} U_k$, and K is compact, so $\exists k_1,\dots,k_n \in K$ s.t.

$$K \subseteq U_{k_1} \cup \dots \cup U_{k_n} =: U. \text{ Let } V = V_{k_1} \cap \dots \cap V_{k_n}. \text{ Both } U \text{ and } V \text{ are open \& disjoint, \&}$$

$$K \subseteq U,\ x \in V. \Rightarrow X-K \text{ is open.} \quad \square$$

Remark We have actually shown more: given a compact K & $x \in X-K$, we can separate them with open sets.

Cor: A compact Hausdorff space is T_3.

Pf: If $V \subseteq X$ is closed, then X compact $\Rightarrow V$ compact. V compact & $x \in X - V$, then the previous prop $\Rightarrow \exists$ disjoint opens $U \supseteq V$, $U' \ni x$. □

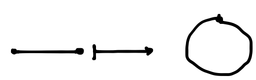
Prop A compact Hausdorff space is T_4.

Pf: Let V_1, V_2 be disjoint, closed sets. X compact $\Rightarrow V_1, V_2$ compact. $\forall v \in V_2$, let $U_v \supseteq V_1$, $U'_v \ni v$ be disjoint opens. $V_2 \subseteq \bigcup_{v \in V_2} U'_v$, and V_2 compact $\Rightarrow \exists v_1, \dots, v_n \in V_2$ s.t. $V_2 \subseteq U'_{v_1} \cup \dots \cup U'_{v_n} =: U'$. Let $U = U_{v_1} \cap \dots \cap U_{v_n} \supseteq V_1$, then U, U' are open & disjoint. □

Prop: If X is compact & Y is Hausdorff, then any continuous $f: X \longrightarrow Y$ is closed.

Pf: If $V \subseteq X$ is closed $\Rightarrow V$ is compact $\Rightarrow f(V)$ is compact $\underset{Y \text{ Hausdorff}}{\Longrightarrow} f(V)$ closed. □

Cor A continuous bijection from a compact space to a Hausdorff space is a homeo.

Ex: Let $I = [0,1]$. Define $\sim$ by $x \sim y$ iff $(x = y)$ or $(\{x,y\} \subseteq \{0,1\})$. So $I/\sim$ (which is compact) is . Let $e: I/\sim \longrightarrow S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}$ be $t \longmapsto e^{2\pi i t}$. This is a continuous bijection $\Rightarrow$ homeomorphism! □