Def: $Y \subseteq X$ is compact if for every open cover $\{U_i\}_{i \in I}$ of $Y$, there is a finite set $\{i_1, \ldots, i_n\}$ s.t. $Y \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$.

The slogan is that "every cover has a finite subcover."

Ex: 1) Any finite set is compact: (there are only finitely many things to check)

2) Any space with the cofinite topology is compact: any non-empty open misses finitely many points, so we only need finitely more.

Pap: Let $X$ be compact & let $f: X \to Y$ be continuous. Then $f(X) \subseteq Y$ is compact.

PI: Let $\{U_i\}_{i \in I}$ be an open cover of $f(X)$ in $Y$. Then since $f$ is continuous, $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of $f^{-1}(f(X))$ and hence of $X$. Since $X$ is compact, for some $\{i_1, \ldots, i_n\} \subseteq I$,

$X \subseteq f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_n}) \Rightarrow f(X) \subseteq f(f^{-1}(U_{i_1})) \cup \cdots \cup f(f^{-1}(U_{i_n})) \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$. Hence we have a finite subcover.

Cor: If $X$ is compact, then so is any quotient.

Pap: If $X$ is compact & $V \subseteq X$ is closed, then $V$ is compact.

PI: Let $\{U_i\}_{i \in I}$ be an open cover of $V$. Since $V$ is closed, $X-V$ is open, and hence $\{X-V\} \cup \{U_i\}_{i \in I}$ is an open cover of $X$. Since $X$ is compact, this has a finite subcover $\{X-V\} \cup \{U_{i_1}, \ldots, U_{i_n}\}$ (without loss of generality, we can assume $X-V$ is in the subcover, since adding it in doesn't change "finite"). Then $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcover of $\{U_i\}_{i \in I}$, and $V$ is compact.

We have a converse to this when $X$ is Hausdorff.

Pap: If $X$ is Hausdorff, then if $K \subseteq X$ is compact, then $K$ is closed.

PI: Let $K \subseteq X$ be compact & let $x \in X-K$. Since $X$ is Hausdorff, $V := K, K, V$ disjoint open $U_k := \{x \} \cup V_k$. Now $K \subseteq \bigcup_{k \in K} U_k$, and $K$ is compact, so $\exists i_1, \ldots, i_n \in K$ s.t. $K \subseteq U_{i_1} \cup \cdots \cup U_{i_n} = U$. Let $V := V_{i_1} \cup \cdots \cup V_{i_n}$. Both $U$ and $V$ are open & disjoint, & $K \subseteq U, x \in V \Rightarrow X-K$ is open.

Remark: We have actually shown more: given a compact $K \subseteq x \in X-K$, we can separate them with open sets.
Sec: A compact Hausdorff space is $T_3$.

PF: If $V \subseteq X$ is closed, then $X$ compact $\Rightarrow$ $V$ compact. $V$ compact $\Rightarrow \exists x \in X - V$, then the previous prop $\Rightarrow \exists$ disjoint opens $U \supseteq V$, $U' \ni x$. □

Prop: A compact Hausdorff space is $T_4$.

PF: Let $V_1, V_2$ be disjoint, closed sets. $X$ compact $\Rightarrow V_1, V_2$ compact $\forall \in V_1$, let $U_\alpha \supseteq V_1$, $U_\alpha \supseteq V_2$ be disjoint opens. $V_2 \subseteq \bigcup_{\alpha \in V_1} U_\alpha$, and $V_2$ compact $\Rightarrow \exists V_3, \ldots, V_n \in V_1$ s.t. $V_2 \subseteq U_{V_1} \cup \ldots \cup U_{V_n} =: U_3$. Let $U = U_{V_1} \cap \ldots \cap U_{V_n} = V_1$, then $U, U'$ are open & disjoint. □

Prop: If $X$ is compact & $Y$ is Hausdorff, then any continuous $f : X \to Y$ is closed.

PF: If $V \subseteq X$ is closed $\Rightarrow V$ is compact $\Rightarrow f(V)$ is compact $\Rightarrow f(V)$ closed. □

Cor: A continuous bijection from a compact space to a Hausdorff space is a homeo.

Ex: Let $I = [0, 1]$. Define $\sim$ by $x \sim y$ iff $(x, y)$ or $[x, y]$ or $[0, 1]$. So $I / \sim$ (which is compact) is $\sim$. Let $e : I / \sim \to S' = \{ z \in \mathbb{C} | d(z, 0) \}\subseteq \mathbb{C}$ be $t \mapsto e^{2\pi t}$. This is a continuous bijection $\Rightarrow$ homeomorphism! □