Lemma (Uryshon's Lemma) If X is normal, then given $E, F \subseteq X$, closed & disjoint, then $\exists f : X \to [0,1]$ continuous s.t. $f(E) \subseteq \{0\}$, $f(F) \subseteq \{1\}$.

Pf By iteratively applying the lemma from last time, build for each dyadic rational r an open set U_r s.t.

① $E \subseteq U_r$, $U_r \subseteq F$ & ② $r < s \Rightarrow \overline{U}_r \subseteq U_s$.

Let $f : X \to [0,1]$ be $f(x) = \begin{cases} 0 & x \in U_r \ \forall r \\ \sup\{r \mid x \notin U_r\} \end{cases}$.

Since $\forall r$, $E \subseteq U_r \subseteq F$, $f(E) = \{0\}$ & $f(F) = \{1\}$. We have to show f is continuous.

Let $x \in X$. Assume $f(x) \in (0,1)$, since other cases are similar & simpler.

For all $\epsilon > 0$, choose r, s dyadic s.t. $f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$. Can do this since they are dense. By construction, $\forall r < f(x)$, $x \notin U_r$. Since $r < r' < f(x) \Rightarrow \overline{U}_r \subseteq U_{r'}$, we conclude $x \in \overline{U}_r$. Since $s > f(x)$, $x \in U_s$, and hence $x \in U_s - \overline{U}_r$ (an open). If $y \in U_s - \overline{U}_r$, then $y \in U_r \subseteq \overline{U}_r \Rightarrow f(y) \geq r$ & $y \in U_s \Rightarrow f(y) \leq s$. $\Rightarrow f(U_s - \overline{U}_r) \in B_\epsilon(f(x))$. □

Remark All finite closed intervals are homeomorphic, so can replace $[0,1]$ by any $[a,b]$.

Now we need some metric properties of functions to $\mathbb{R}$.

Def Let S be a set & let X be a metric space. Define d_∞ on $F(S,X)$ by

$$d_\infty(f,g) = \sup_{s \in S} \{ \min(1, d(f(s), g(s))) \}.$$

Prop d_∞ is a metric on $F(S,X)$.

Def A sequence of functions f_n <u>converges uniformly</u> to f if $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $s \in S$,

$$d(f_n(s), f(s)) < \epsilon.$$

Prop: $[f_n] \longrightarrow f$ iff f_n converges uniformly to f.

Prop If X is complete, then so is $F(X,S)$ with d_∞.

Pf: Let f_n be a Cauchy sequence in $F(x,S)$: $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n,m \geq N$, $\forall s \in S$, $d(f_n(s), f_m(s)) < \epsilon$. In particular, $\forall s \in S$, $[f_n(s)]$ is a Cauchy sequence in $X \Rightarrow$ has a limit. Let $f(s) = \lim f_n(s)$.

Then we need to show $f_n \longrightarrow f$. Let $\epsilon > 0$. Then by Cauchy, $\exists N$ s.t. $\forall n,m \geq N$, $s \in S$,

$$d(f_n(s), f_m(s)) < \epsilon. \quad \text{Now} \quad d(f_n(s), f(s)) \leq d(f_n(s), f_m(s)) + d(f_m(s), f(s)) \leq \epsilon + d(f_m(s), f(s)).$$

$\Rightarrow d(f_n(s), f(s)) \leq \epsilon$. $\square$

Prop If f_n is a seq. of continuous functions converging uniformly to f, then f is continuous.

Pf: Let $\epsilon > 0$, and let N be s.t. $\forall n \geq N$, $y \in X$, $d(f_n(y), f(y)) < \epsilon/3$. Since f_n is continuous at x, $\exists\ U \ni x$ s.t. $\forall y \in U$, $d(f_n(y), f_n(x)) < \epsilon/3$. Now $\forall y \in U$,

$$d(f(y), f(x)) \leq d(f(y), f_n(y)) + d(f_n(y), f_n(x)) + d(f_n(x), f(x)) < 3 \cdot \epsilon/3 = \epsilon.$$

f is continuous iff it is continuous at every point.

Thm (Tietze Extension Thm) Let X be normal & $Y \subseteq X$ closed. If $f: Y \longrightarrow \mathbb{R}$ is continuous & bounded, then $\exists\ \tilde{f}: X \longrightarrow \mathbb{R}$, continuous, bounded, and $\tilde{f}(y) = f(y)$ $\forall y \in Y$.

Pf: Let $c_0 = \sup_{y \in Y} \{|f(y)|\} < \infty$ (f is bounded). I.e. $f(Y) \subseteq [-c_0, c_0]$. Let $E_0 = f^{-1}([-c_0, -c_0/3])$, $F_0 = f^{-1}([c_0/3, c_0])$. These are closed & disjoint. Urysohn $\Rightarrow \exists\ g_0: X \longrightarrow [-c_0/3, c_0/3]$ s.t. $g_0(E_0) = -c_0/3$, $g_0(F_0) = c_0/3$. $\Rightarrow$ (i) $|g_0| \leq c_0/3$ & (ii) $|f - g_0| \leq \frac{2}{3} c_0$ on Y.

Let $f_1 = f - g_0$. By induction, we can produce a sequence of functions $g_n: X \longrightarrow \mathbb{R}$ s.t.

① $|g_n(x)| \leq \frac{2^n}{3^{n+1}} c_0$ & ② $|f - g_0 - \dots - g_n| \leq \left(\frac{2}{3}\right)^{n+1} c_0$, (Simply let $f_n = f - g_0 - \dots - g_{n-1}$ & apply the same argument. Now if $h_n := g_0 + \dots + g_n$, then

ⓐ $[h_n]$ is Cauchy: If $n > m$, then $|h_n - h_m| = \left|\sum_{i=m+1}^{n} g_i\right| \leq \sum_{i=m+1}^{n} |g_i| \leq \left(\frac{2^{m+1}}{3^{m+2}} c_0 + \dots + \frac{2^n}{3^{n+1}} c_0\right)$

$$= \frac{2^{m+1}}{3^{m+2}} c_0 \left(1 + \dots + \left(\frac{2}{3}\right)^{n-m-1}\right) \leq \left(\frac{2}{3}\right)^{m+1} c_0 \longrightarrow 0.$$

ⓑ The limit h agrees with f on Y: $[f - h_n] \longrightarrow 0$ on Y, since $|f - h_n| \leq \left(\frac{2}{3}\right)^{n+1} c_0 \longrightarrow 0$.