Prop If X is Hausdorff, then limits of sequences in X are unique.

Pf: $\left.\begin{array}{l} x \text{ is a limit of } [x_n] \text{ iff } \forall\, U \ni x,\ \exists N \text{ s.t. } n \geq N \Rightarrow x_n \in U. \\ y \text{ is a limit of } [x_n] \text{ iff } \forall\, V \ni y,\ \exists M \text{ s.t. } m \geq M \Rightarrow x_m \in V \end{array}\right\} \Rightarrow \begin{array}{l} \forall n \geq \max(N,M), \\ x_n \in U \cap V \Rightarrow \\ U \cap V \neq \emptyset. \end{array}$

So every neighborhood of x intersects every neighborhood of y non-trivially. X Hausdorff $\Rightarrow x = y$. $\square$

Def X is regular if whenever $x \in X$, $V \not\ni x$ closed, have disjoint opens U_1 & U_2 s.t.

$x \in U_1$, $V \subseteq U_2$.

There is a straightforward specialization to Hausdorff if we know points are closed.

Def X is T_3 if it is regular and T_1.

Def X is normal if $\forall\, V_1, V_2$ closed, $V_1 \cap V_2 = \emptyset$, $\exists U_1, U_2$ open, $U_1 \cap U_2 = \emptyset$ & $V_i \subseteq U_i$.

Def X is T_4 if it is normal & T_1.

Prop $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

Lemma Metric spaces are T_4.

Pf Let V_1, V_2 be disjoint closed sets. Since $U_i' = X - V_i$ is open, $\forall\, x \in V_1$, $\exists\, r_x$ s.t. $B_{r_x}(x) \subseteq U_2'$ & $\forall y \in V_2$, $\exists\, r_y$ s.t. $B_{r_y}(y) \subseteq U_1'$. Let $U_1 = \bigcup_{x \in V_1} B_{r_x/2}(x)$ and sim. for $U_2 = \bigcup_{y \in V_2} B_{r_y/2}(y)$. If $z \in U_1 \cap U_2$, then $\exists\, x \in V_1$ s.t. $z \in B_{r_x/2}(x)$ & $\exists\, y \in V_2$ s.t. $z \in B_{r_y/2}(y)$.

$\Rightarrow$ $d(x,y) \leq d(x,z) + d(z,y) < \frac{r_x}{2} + \frac{r_y}{2} \leq \max(r_x, r_y)$. $\Rightarrow$ either $x \in B_{r_y}(y)$ or $y \in B_{r_x}(x)$, a contradiction. $\square$

Lemma X is normal iff $\forall\, \underset{\text{closed}}{V} \subseteq \underset{\text{open}}{U}$, $\exists\, U'$, open s.t. $V \subseteq U' \subseteq \overline{U'} \subseteq U$.

Pf: $\Rightarrow$) Let $V \subseteq U$, and let $F = X - U$. This is closed and disjoint from V. X normal $\Rightarrow \exists\, U', U''$ open s.t. $V \subseteq U'$, $F \subseteq U''$, $U' \cap U'' = \emptyset$. The latter implies $U' \subseteq X - U''$, which is closed, so $V \subseteq U' \subseteq \overline{U'} \subseteq X - U'' \subseteq X - F = U$.

$\Leftarrow$) Let E & F be closed and disjoint. Then $E \subseteq X - F = U_1$, open. $\Rightarrow \exists\, U'$ s.t. $E \subseteq U' \subseteq \overline{U'} \subseteq U$. Let $U_1 = X - \overline{U'} \supseteq X - U = F$. Since $U' \subseteq \overline{U'}$, $U' \cap U_1 = \emptyset$. $\square$

<u>Lemma</u> (Uryshon) If X is normal & E, F are closed & disjoint, then $\exists$ continuous $f: X \to [0,1]$ s.t. $f(E) \subseteq \{0\}$, $f(F) \subseteq \{1\}$.

<u>Pf</u>: Let $U = X - F$. Then $E \subseteq U$, and the previous lemma gives us a $U_{1/2}$, open, s.t.

$$E \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U.$$

This has the same form, so we can iterate:

$E \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq U$, etc. Continuing in this way gives an open set U_r for every $r \in (0,1)$ of the form $\frac{a}{2^b}$ ("dyadic rationals") with the property that if $r < s$, then $U_r \subseteq \overline{U}_r \subseteq U_s$, $E \subseteq U_r$ $\forall r$ & $U_s \cap F = \emptyset$ $\forall s$. Start here on Friday.