We have talked much less about sequences because they can be very poorly behaved: a sequence can have many limits.

Ex: $\mathbb{Z}$ w/ co-finite topology: $\{1,2,\dots\}$ converges to every point.

Ex: Let $X = \mathbb{R}\times\{0,1\}$ & define an equivalence relation $\sim$ by

$(x,\epsilon)\sim(y,\epsilon')$ iff $\left(x=y,\ \epsilon=\epsilon' \text{ or } x=y\neq 0\right)$. Let $\overline{X} = X/\sim$. Then any sequence that has limit zero in $\mathbb{R}$ has 2 limits: $[(0,0)]$ and $[(0,1)]$

We control this with "separation axioms."

Def A space X is T_0 if $\forall x\neq y\in X$, $\exists U\in\mathcal{T}_X,\ x\in U\subseteq X-\{y\}$ or $\exists V\in\mathcal{T}_X,\ y\in V\subseteq X-\{x\}$.

Ex: $\{a,b\}$ with interesting topology is T_0.

Def A space X is T_1 if $\forall x\in X$, $\{x\}$ is closed.

Prop X is T_1 iff $\forall x\neq y$, $\exists U\in\mathcal{T}_X$ s.t. $x\in U\subseteq X-\{y\}$.

Pf: $\Rightarrow$) Since $\{y\}$ is closed, $U=X-\{y\}$ is open & works.

$\Leftarrow$) This is the condition that $U-\{y\}$ is open. $\square$

Def X is T_2 or Hausdorff if $\forall x\neq y$, $\exists U,V\in\mathcal{T}_X$ s.t. $x\in U, y\in V$, $U\cap V=\emptyset$.

Prop If X is Hausdorff, then limits are unique.

Pf: Let x and x' be limits for $[x_n]$. Then $\forall\ U,V\in\mathcal{T}_X$, $x\in U$, $x'\in V$, $\exists N$ s.t. $\forall n\geq N$, $x_n\in U$ & $x_n\in V$. In particular, $U\cap V\neq\emptyset$, and since X is Hausdorff, $x=x'$. $\square$

Ex: Every metric space is Hausdorff.

If $S\subseteq X$ & X is Hausdorff, then S is Hausdorff.

Prop If X & Y are Hausdorff, then so is $X\times Y$.

Pf If $(x_1,y_1)\neq(x_2,y_2)$, then $x_1\neq x_2$ or $y_1\neq y_2$. If the former, find $U_1,U_2\in\mathcal{T}_X$ s.t. $x_i\in U_i$ & $U_1\cap U_2=\emptyset$

Then $(x_1,y_1)\in U_1\times Y$ & $(x_2,y_2)\in U_2\times Y$, & $U_1\times Y\cap U_2\times Y=\emptyset$. The other case is the same. $\square$

Products give another way to describe Hausdorff.

Lemma X is Hausdorff iff $\Delta=\{(x,x)\mid x\in X\}\subseteq X\times X$ is closed.

Pf: The diagonal is closed iff $X-\Delta$ is open iff $\forall\,(x,y)\in X\times X-\Delta$, we can find $U,V\in\tau_X$ s.t. $(x,y)\in U\times V\subseteq X\times X-\Delta$. Now $U\times V\cap\Delta=\{(z,z)\mid z\in U\cap V\}=\Delta|_{U\cap V}$, so $U\times V\subseteq X\times X-\Delta$ iff $U\cap V=\emptyset$. $\square$

Fun application: topological groups!

Prop 1) addition & multiplication are continuous functions $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

2) inversion is a continuous function $\mathbb{R}-\{0\}\longrightarrow\mathbb{R}-\{0\}$.

Also true for $\mathbb{Q},\mathbb{C}$, etc!

Pf: Exercise in metric spaces. $\square$

Cor i) Any polynomial $p(x)=a_nx^n+\ldots+a_0$ is continuous as a function $\mathbb{R}\longrightarrow\mathbb{R}$.

ii) Any rational function $\frac{p}{q}=\dfrac{a_nx^n+\ldots+a_0}{b_mx^m+\ldots+b_0}$ is continuous as a function $\mathbb{R}-\{\text{roots of } q\}\longrightarrow\mathbb{R}$.

Pf: These are composites of continuous functions.

Def A topological group is a group G + a topology on G s.t.

1) Multiplication is continuous: $\cdot : G\times G\longrightarrow G$

2) Inversion is continuous: $(\ )^{-1}: G\longrightarrow G$

3) G is Hausdorff

Prop If G is a group w/ a topology s.t. 1) & 2) hold & G is T_1, then 3) holds.

Pf: Consider
$$f: G\times G\longrightarrow G\times G\longrightarrow G.$$
$$(g,h)\longmapsto(g,h^{-1})\longmapsto g\cdot h^{-1}$$
This is a composite of continuous functions. G T_1 $\Rightarrow\{e\}\subseteq G$ is closed $\Rightarrow f^{-1}(\{e\})=\{(g,h)\mid gh^{-1}=e\}=\Delta$ is closed. $\square$

Examples ① $(\mathbb{R},+)$, $(\mathbb{C},+)$, any normed vector space.

② $(\mathbb{R}^\times, \times)$, $(\mathbb{C}^\times, \times)$

③ $S^1 = \{e^{2\pi i x} \mid x \in \mathbb{R}\} \subseteq \mathbb{C}^\times$.

Def Let $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\} \subseteq M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Prop 1) $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$

2) The mult on $M_n(\mathbb{R})$ is continuous

3) Inversion on $GL_n(\mathbb{R})$ is continuous.

Pf: 1) det is a polynomial in the coordinates $\Rightarrow$ continuous $\Rightarrow$ $\det^{-1}(\mathbb{R}-\{0\}) = GL_n\mathbb{R}$ is open.

2) The $(i,j)^{th}$ coordinate in matrix mult is a polynomial in the coords of the factors.

3) Have a formula for inversion that is $\frac{1}{\det(A)} \cdot \bar{a}_{ij}$, where $\bar{a}_{ij}$ is a det of a sub-matrix.

det is a polynomial & never zero, so $\frac{1}{\det}$ is continuous. $A \mapsto \bar{a}_{ij}$ is also poly, hence cont. □