Prop If $X \xrightarrow{f_1} Y_1, \ldots, X \xrightarrow{f_n} Y_n, \ldots$ is a sequence of functions, then if $S = \tau_X^{f_1} \cup \ldots$, then

(i) $\forall i$, $f_i : X \to Y_i$ is continuous for τ_S &

(ii) if τ is any topology on X s.t. f_i is continuous $\forall i$, then $\tau_S \subseteq \tau$.

Pf: (i) $\forall i$, and $\forall U \in \tau_{Y_i}$, $f_i^{-1}(U) \in \tau_X^{f_i} \subseteq S \subseteq \tau_S$. $\Rightarrow f_i$ is continuous.

(ii) If $f_i : X \to Y_i$ is continuous, then $\forall U \in \tau_{Y_i}$, $f^{-1}(U) \in \tau$. $\Rightarrow \tau_X^{f_i} \subseteq \tau$ $\forall i$, so

$\tau_X^{f_1} \cup \ldots = S \subseteq \tau$. τ is a topology containing $S \Rightarrow \tau \supseteq \tau_S$. □

Let's apply this to products

Def If (X, τ_X) and (Y, τ_Y) are spaces, then the <u>product topology</u> on $X \times Y$ is the coarsest topology s.t. both projections $X \times Y \xrightarrow{\pi_X} X$ & $X \times Y \xrightarrow{\pi_Y} Y$ are continuous.

Prop A basis for the product topology is given by sets of the form $U_1 \times U_2$, $U_1 \in \tau_X$, $U_2 \in \tau_Y$.

Pf: A sub-basis is given by the collections $\{U \times Y \mid U \in \tau_X\} \cup \{X \times V \mid V \in \tau_Y\}$. The basis for the topology is given by finite intersections of these. So consider $(U_1 \times Y) \cap (U_2 \times Y) \cap \cdots \cap (U_n \times Y) \cap (X \times V_1) \cap \cdots \cap (X \times V_m)$

$= \Big(\big(\underbrace{U_1 \cap \cdots \cap U_n}_{U} \big) \times Y \Big) \cap \Big(X \times \big(\underbrace{V_1 \cap \cdots \cap V_m}_{V} \big) \Big) = U \times V$. □

Thm $Z \xrightarrow{f} X \times Y$ is continuous iff $\pi_X \circ f$ & $\pi_Y \circ f$ are.

Pf: $\Rightarrow$) Since π_X & π_Y are continuous, so are $\pi_X \circ f$ & $\pi_Y \circ f$.

$\Leftarrow$) If $\pi_X \circ f$ is continuous, then $\forall U \in \tau_X$, $(\pi_X \circ f)^{-1}(U) = f^{-1}(U \times Y)$ is open. Similarly, $f^{-1}(X \times V)$ is open $\forall V \in \tau_Y$. By the lemma, since $\tau_{X \times Y}^{\pi_X} \cup \tau_{X \times Y}^{\pi_Y}$ is a sub-basis, f is continuous □

Def If X_i, $i \in I$ is a sequence of spaces, then the <u>product topology</u> on $\prod_{i \in I} X_i$ is the smallest topology such that all projection maps $\prod_{i \in I} X_i \xrightarrow{\pi_i} X_i$ are continuous.

Prop A basis for $\prod_{i \in I} X_i$ is given by $\prod_{i \in I} U_i$, $U_i = X_i$ for all but finitely many i.

Pf: A subbase is given by $\prod_{i \in I} U_i$, $U_i = X_i$ for all but one $i \in I$. □

There is an extremely important dual notion.

Def If X is a space & $X \xrightarrow{f} Y$ is a function, then let $\tau_f^Y = \{U \subseteq Y \mid f^{-1}(U) \in \tau_X\}$.

Prop (i) τ_f^Y is a topology on Y.

(ii) $f : (X, \tau_X) \to (Y, \tau_f^Y)$ is continuous

(iii) If $f:(X,\tau_X)\longrightarrow(Y,\tau)$ is continuous, then $\tau\subseteq\tau_f^Y$.

Pf: (i) $\emptyset=f^{-1}(\emptyset)\in\tau_X\Rightarrow\emptyset\in\tau_f^Y$. $X=f^{-1}(Y)\in\tau_X\Rightarrow Y\in\tau_f^Y$. If $\{U_i\}_{i\in I}\subseteq\tau_f^Y$, then have to check that $f^{-1}\left(\bigcup_{i\in I}U_i\right)\in\tau_X$. But $f^{-1}\left(\bigcup_{i\in I}U_i\right)=\bigcup_{i\in I}\underbrace{f^{-1}(U_i)}_{\in\,\tau_X \text{ by assumption}}\in\tau_X$.

$U_1,\dots,U_n\in\tau_f^Y\leftrightarrow f^{-1}(U_1),\dots,f^{-1}(U_n)\in\tau_X$, so

$$f^{-1}(U_1\cap\cdots\cap U_n)=f^{-1}(U_1)\cap\cdots\cap f^{-1}(U_n)\in\tau_X\leftrightarrow U_1\cap\dots\cap U_n\in\tau_f^Y.$$

(ii) $U\in\tau_f^Y\leftrightarrow f^{-1}(U)\in\tau_X$, so f is continuous.

(iii) If f is continuous, then $U\in\tau\Rightarrow f^{-1}(U)\in\tau_X\Leftrightarrow U\in\tau_f^Y$. So $\tau\in\tau_f^Y$. $\square$

Def If $X\xrightarrow{q}Y$ is surjective, then the quotient topology on Y is τ_q^Y.

If $\sim$ is an equivalence relation on X, then the quotient topology on $X/\sim$ is $\tau_q^{X/\sim}$, where

$$q: X\longrightarrow X/\sim,\quad x\longmapsto[x]$$ is the quotient map.

Thm: Let $f:X\to Y$ be such that if $x\sim y$, then $f(x)=f(y)$. Then $f=\bar{f}\circ q$ for some $\bar{f}:X/\sim\,\to Y$ and f is continuous iff $\bar{f}$ is.

Pf: $\Leftarrow$) If $\bar{f}$ is cont, then since q is continuous, $f=\bar{f}\circ q$ is.

$\Rightarrow$) Let $U\in\tau_Y$. Then $\bar{f}^{-1}(U)$ is open iff $q^{-1}(\bar{f}^{-1}(U))=(\bar{f}\circ q)^{-1}(U)=f^{-1}(U)$ is open. $\square$