Last time, we looked at a topology induced by a map. Now we'll consider several.

Observation: If $f_1 : X \to Y_1$ & $f_2 : X \to Y_2$, then $f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$ need not be of the form $f_1^{-1}(U)$ or $f_2^{-1}(U)$. So our argument must be modified.

Prop If $t \subseteq P(X)$ is any collection of subsets, then there is a minimal topology τ^t s.t. $t \subseteq \tau^t$.

Pf: Let $\tau^t = \bigcap_{t \subseteq \tau} \tau$. $\square$

We want a better way to describe this. Work down in 2 steps.

Def $\mathcal{B} \subseteq \tau$ is a basis for the topology if $\forall U \in \tau$, $\exists \{U_i\}_{i \in I} \subseteq \mathcal{B}$ s.t. $U = \bigcup_{i \in I} U_i$.

Prop $\mathcal{B}$ is a basis for τ iff $\forall U \in \tau$ & $x \in U$, $\exists U' \in \mathcal{B}$ s.t. $x \in U' \subseteq U$.

Pf: Same as for metric spaces.

So when is a collection a basis for some topology?

Prop $\mathcal{B}$ is a basis for a topology iff $\forall U, V \in \mathcal{B}$, $x \in U \cap V$, $\exists W \in \mathcal{B}$ s.t. $x \in W \subseteq U \cap V$, & $\mathcal{B}$ covers.

Pf: If $\mathcal{B}$ is a basis, then U, V are open $\Rightarrow U \cap V$ is open. Previous prop $\Rightarrow \exists$ such a W.

$\Leftarrow$) We need to show that $\tau = \{U = \bigcup_{\{U_i\} \subseteq \mathcal{B}} U_i\}$ is a topology. This is closed under arbitrary unions

$\emptyset = \bigcup_{\emptyset}$, so this is in τ & $\mathcal{B}$ covering $\Rightarrow X$ is. We need only show closure under finite intersections. By induction, it suffices to show $\left(\bigcup_{i \in I} U_i\right) \cap \left(\bigcup_{j \in J} V_j\right) = \bigcup_{\substack{i \in I \\ j \in J}} (U_i \cap V_j) \in \tau$. So in turn, it suffices to show $U_i \cap V_j \in \tau$. However, for all $x \in U_i \cap V_j$, $\exists W_x \in \mathcal{B}$ s.t. $x \in W_x \subseteq U_i \cap V_j$.

$\Rightarrow U_i \cap V_j \subseteq \bigcup_{x \in U_i \cap V_j} W_x \subseteq U_i \cap V_j$, and $U_i \cap V_j \in \tau$. $\square$

Prop If $\mathcal{B}_1$ & $\mathcal{B}_2$ are bases, then $\tau_{\mathcal{B}_1} \subseteq \tau_{\mathcal{B}_2}$ iff $\forall U \in \mathcal{B}_1$ $U \in \tau_{\mathcal{B}_2}$.

Pf $\Rightarrow$) Since $\mathcal{B}_1 \subseteq \tau_{\mathcal{B}_1} \subseteq \tau_{\mathcal{B}_2}$, this is immediate.

$\Leftarrow$) Let $U \in \tau_{\mathcal{B}_1}$. Then $U = \bigcup_{i \in I} U_i$, $U_i \in \mathcal{B}_1$ $\forall i \in I$. But $\mathcal{B}_1 \subseteq \tau_{\mathcal{B}_2} \Rightarrow U$ is a union of opens in $\tau_{\mathcal{B}_2} \Rightarrow U$ is open in $\tau_{\mathcal{B}_2}$. $\Rightarrow \tau_{\mathcal{B}_1} \subseteq \tau_{\mathcal{B}_2}$. $\square$

Def A <u>sub-basis</u> for a topology τ is any collection $S \subseteq \tau$ s.t. $\{X\} \cup \{U_1 \cap \dots \cap U_n \mid {n \in \mathbb{N} \atop U_i \in S}\}$ is a basis for τ.

Prop Any set $S \subseteq P(X)$ is a sub-basis for the topology generated by S.

Pf: Exercise.

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Lem Let $f: X \to Y$, and let S be a subbasis for τ_Y. Then f is continuous iff $\forall U \in S$, $f^{-1}(U) \in \tau_X$.

Pf: $\Rightarrow$) $S \subseteq \tau_Y$, so f continuous implies this.

$\Leftarrow$) Any open in Y is of the form $\bigcup_i U_i$, where $U_i = U_{i1} \cap \cdots \cap U_{ij}$, all $U_{ij} \in S$. So

$$f^{-1}\Big(\bigcup_i U_i\Big) = \bigcup_i f^{-1}(U_i) = \bigcup_i f^{-1}(U_{i1} \cap \cdots \cap U_{ij}) = \underbrace{\bigcup_i \underbrace{f^{-1}(U_{i1}) \cap \cdots \cap f^{-1}(U_{ij})}_{\text{open}}}_{\text{open.}}$$

all are open!

$\square$

So this answers our first question: the topology generated by S is all unions of finite intersections of elements of S.