<u>Def</u> Let X and Y be topological spaces. A function $f: X \to Y$ is <u>continuous</u> if $\forall U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open.

This is <u>the</u> definition that always works. The homework shows a way the limit version can be harder to apply

<u>Remark</u>: Any map $f: X \to Y$ induces a map $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$, $S \mapsto f^{-1}(S)$. f is continuous if $f^{-1}(\tau_Y) \subseteq \tau_X$.

<u>Prop</u> Let τ_1 & τ_2 be two topologies on X. Then $\mathrm{Id}: (X, \tau_1) \to (X, \tau_2)$ is continuous iff $\tau_2 \subseteq \tau_1$.

<u>Pf</u>: Id is continuous iff $\forall U \in \tau_2$, $(\mathrm{Id})^{-1}(U) = U \in \tau_1$. $\square$

<u>Def</u>: We say τ_2 is <u>coarser</u> than τ_1 & τ_1 is <u>finer</u> than τ_2 if $\tau_2 \subseteq \tau_1$.

<u>Prop</u> For any space X and any indiscrete Y, all functions $X \to Y$ are continuous.

For any discrete X and any Y, all functions are continuous.

<u>Ex</u>: Let $Y = \{a, b\}$ w/ $\tau = \{\emptyset, \{a\}, Y\}$. Then $f: X \to Y$ is continuous iff $f^{-1}(\{a\})$ is open.

In particular $\mathrm{Map}(X,Y) \to \tau_X$, $f \mapsto f^{-1}(\{a\})$, $\tau_X \to \mathrm{Map}(X,Y)$, $U \mapsto \begin{cases} U \mapsto a \\ X - U \mapsto b \end{cases}$ are inverses.

<u>Prop</u> If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are continuous, then so is $g \circ f: X \to Z$.

<u>Pf</u>: Same as in the metric case.

We can use maps to <u>define</u> a topology on a set.

<u>Def</u> Let X be a space and let $S \subseteq X$. The <u>subspace topology</u> on S is the coarsest topology on S s.t. the inclusion $S \subseteq X$ is continuous.

Let's unpack this. ① If $S \subseteq X$, then we have a function $\iota_S: S \to X$ defined by $\iota_S(s) = s$.

② Coarsest here means smallest collection of open sets that works (by Prop above, the discrete top always would work).

<u>Prop</u> Let X be a set and let (Y, τ_Y) be a space. Let $f: X \to Y$ be a function. Then

(i) $\tau_X^f = \{f^{-1}(U) \mid U \in \tau_Y\}$ is a topology on X &

(ii) f is continuous wrt τ_X^f & if τ_X is any top. on X for which f is cont, then $\tau_X^f \subseteq \tau_X$.

Pf: (i) Since $\emptyset$ & $Y \in \tau_Y$, $\emptyset = f^{-1}(\emptyset)$ & $X = f^{-1}(Y) \in \tau_X$. If $\{f^{-1}(U_i)\}_{i\in I} \subseteq \tau_X^f$, then

$$\bigcup_{i\in I} f^{-1}(U_i) = f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \in \tau_X^f \qquad \Big(\{U_i\} \subseteq \tau_Y \Rightarrow \bigcup U_i \in \tau_Y\Big)$$

$$f^{-1}(U_1)\cap\dots\cap f^{-1}(U_n) = f^{-1}(U_1\cap\dots\cap U_n) \in \tau_X^f \qquad (U_1,\dots,U_n \in \tau_Y \Rightarrow U_1\cap\dots\cap U_n \in \tau_Y).$$

(ii) f is visibly continuous. If f is continuous for τ_X, then $\forall U \in \tau_Y$, $f^{-1}(U) \in \tau_X$. However, every element of τ_X^f is of this form. $\Rightarrow \tau_X^f \subseteq \tau_X$. $\square$

Cor If $S \subseteq X$, then $U \subseteq S$ is open iff $\exists\, U' \in \tau_X$ with $U' \cap S = U$.

Pf: $\iota_S^{-1}(U') = U' \cap S$. $\square$

Thm Let $Y \xrightarrow{f} X$ be a function such that $\mathrm{Im}(f) \subseteq S$. Write $\tilde{f}: Y \to S$ for the lift. Then f is continuous iff $\tilde{f}$ is.

$$\begin{tikzcd} Y \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & S \arrow[d, hook, "\iota_S"] \\ & X \end{tikzcd}$$

"Universal property of a subspace"

Pf: If $\tilde{f}$ is continuous, then $f = \iota_S \circ \tilde{f}$ is.

Otherwise, let $U \subseteq S$ be open. By the above, $U = \iota_S^{-1}(U')$ some open U' in X.

Now $\tilde{f}^{-1}(U) = \tilde{f}^{-1}(\iota_S^{-1}U') = (\iota_S \circ \tilde{f})^{-1}(U') = f^{-1}(U')$ is open (f is continuous). $\square$

Remark Applying this to $Y = S$ with possibly different topologies, we see again that τ_S is the coarsest possible.