We now get to the heart of the course: topological spaces!

**Def**: A topology on $X$ is a collection $\tau$ of subsets of $X$ s.t.

① $\phi, X \in \tau$ ② If $\forall i \in I$, $U_i \in \tau$, then $\bigcup U_i \in \tau$. ③ If $U_1, \dots, U_n \in \tau$, then $U_1 \cap \dots \cap U_n \in \tau$.

Same ideas as a metric space, just without the metric. We are teasing apart what matters.

**Examples** If $X$ is a metric space, then $\tau = \{U \mid U \text{ is open}\}$ is a topology.

2) If $X$ is any set, then $\tau = \{\phi, X\}$ is a topology (the indiscrete topology)

3) If $X$ is any set, then $\tau = \{Y \mid Y \subseteq X\}$ is a topology (the discrete topology)

4) $X = \{a, b\}$, $\tau = \{\phi, \{a\}, X\}$.

**Def** A set $V$ is closed if $X - V$ is open.

**Prop** A topology on $X$ is equivalently a collection of subsets of $X$ $\tau^c$ s.t.

① $X, \phi \in \tau^c$; ②' If $\forall i \in I$, $V_i \in \tau^c$, then $\bigcap_{i \in I} V_i \in \tau^c$ ; ③' If $V_1, \dots, V_n \in \tau^c$, then $V_1 \cup \dots \cup V_n \in \tau^c$.

**Pf** $U \longmapsto X - U$ gives us an identification $\tau \longleftrightarrow \tau^c$. □

**Def** An open neighborhood of $x \in X$ is an open set $U$ s.t. $x \in U$.

**Def** $x \in S$ is an interior point if $\exists$ an open neighborhood $U \subseteq S$.

The interior of $S$ is $\{s \in S \mid s \text{ is an interior point}\}$.

**Prop** The interior of $S$ is open. $S$ is open iff $S = \mathrm{int}(S)$.

**Pf**: If $s \in \mathrm{int}(S)$, then $\exists\ U_s \subseteq S$, $U_s$ open. $U_s$ is an open neighborhood for all $s' \in U_s$, So $U_s \subseteq \mathrm{int}(S)$. Then $\mathrm{int}(S) \subseteq \bigcup_{s \in \mathrm{int}(S)} U_s \subseteq \mathrm{int}(S)$. For the second part, if $S$ is open, then $S$ is an open neighborhood of all of its points. □

**Def** The closure of $S$ is the comp of the interior of the comp of $S$:

$$\overline{S} = X - \mathrm{int}(X - S).$$

**Prop**: $\overline{S}$ is closed & $\overline{(\overline{S})} = \overline{S}$.

**Pf**: $\mathrm{int}(X - S)$ is open, so $\overline{S}$ is closed. If $V$ is closed, then $X - V$ is open, so $\overline{V} = X - \mathrm{int}(X - V) = X - (X - V) = V$. □

Def A point $x \in X$ is adherent to $S$ if $\forall\, U \ni x$, open, $U \cap S \neq \emptyset$.

Prop $\overline{S} = \{x \in X \mid x \text{ is adherent to } S\}$

Pf: $z \in \overline{S}$ iff $z \in X - \text{int}(X-S)$ iff $\forall\, U \ni z$, $U \not\subset \text{int}(X-S)$ iff $\forall\, U \ni z$, $U \not\subset X-S$ iff $U \cap S \neq \emptyset$. $\square$

Def $x \in X$ is a boundary point for $S \subseteq X$ if $x \in \overline{S} \cap \overline{(X-S)}$. The boundary of $S$ is

$$\partial S = \overline{S} \cap \overline{(X-S)}.$$

Prop: $\overline{S} = \text{int}(S) \cup \partial S$

Pf: $\supseteq$) Both $\text{int}(S)$ & $\partial S$ are visibly in $\overline{S}$. $\subseteq$) If $z \in \overline{S}$, then either ① there is an open neighborhood $U$ s.t. $U \subseteq S$ or ② there is no neighborhood satisfying this, ie $\forall\, U \ni z$, $U \cap (X-S) \neq \emptyset$. ① means $z$ is interior. ② means $z \in \overline{X-S}$, and hence $\partial S$. $\square$