MATH 121 – MIDTERM I

APRIL 20^{TH}, 2016

Problem 1. *Give the definitions of the following terms:*

(1) *Compact* $S \subseteq X$ is compact if for every open cover $\{U_i\}_{i \in I}$ of S, there is a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$

(2) *Totally Bounded* X is totally bounded if $\forall \epsilon > 0$, $\exists\ x_1, \dots, x_n \in X$ s.t. $X = B_\epsilon(x_1) \cup \dots \cup B_\epsilon(x_n)$.

(3) *Bounded* X is bounded if $\exists\ r$ s.t. $\forall x, y \in X$, $d(x,y) < r$.
(OR $\exists\ r > 0$, $x \in X$ s.t. $X \subseteq B_r(x)$)

(4) *Continuous* $f : X \to Y$ is continuous if $\forall$ open $V \subseteq Y$, $f^{-1}(V)$ is open
(equivalently: If $[x_n] \to x$ in X, then $[f(x_n)] \to f(x)$ in Y OR
$\forall \epsilon > 0$ & $x \in X$, $\exists\ \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.)

Problem 2. *Let X be a metric space.*

(1) *Show that if $x \in X$, then $\{x\}$ is a closed subset of X.*

Argument I: Part (2) shows $X - \{x\}$ is open.

Argument II: Let $y \in X - \{x\}$ & let $r = d(x,y) > 0$. Then $B_r(y) \subseteq X - \{x\}$, so $X - \{x\}$ is open.

Argument III: There is a unique sequence in $\{x\}$, namely $x_n = x$ $\forall n$. This visibly converges to x. So $\{x\}$ also has all limit points.

(2) *Show that if $x \neq y \in X$, then there are open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Let $r = d(x,y) > 0$. Then if $z \in B_{r/2}(x) \cap B_{r/2}(y)$, then $r = d(x,y) \leq d(x,z) + d(y,z) < r$, a contradiction. $\Rightarrow U = B_{r/2}(x)$ & $V = B_{r/2}(y)$ work.

Problem 3. *If A is a subset of X, then define a function*

$$d_A\colon X \to \mathbb{R}$$

by $d_A(x) = \inf_{a\in A}\{d(x,a)\}$.

Show that $d_A(x) = 0$ if and only if $x \in \overline{A}$.

$d_A(x) = 0$ iff $\forall \epsilon > 0$, $\exists a \in A$ s.t. $d(a,x) < \epsilon$: (If for some ϵ, $d(a,x) \geq \epsilon$ $\forall a \in A$ then $d_A(x) \geq \epsilon$).

For each $n > 0$, choose a_n s.t. $d(a_n, x) < 1/n$.

Then $[a_n]$ is a sequence in A converging to x so $x \in \overline{A}$.

Conversely, if $[a_n] \to x$ is any sequence w/ $a_n \in A$ $\forall n$, then by assumption, $d(a_n, x) \to 0$ & $d_A(x) = 0$.

Problem 4. *Let Y be a subset of a metric space X. Show that the closure of Y is the intersection of all of the closed sets of X that contain Y:*

$$\overline{Y} = \bigcap_{Y\subset V, V=\overline{V}} V.$$

$\supseteq$) $\overline{Y}$ is always a closed set that contains Y, so $\bigcap_{\substack{Y\subseteq V\\ V=\overline{V}}} V \subseteq \overline{Y}$.

$\subseteq$) Let $[y_n]$ be a sequence in Y converging to $y \in \overline{Y}$. If V contains Y and is closed, then V contains the limit of $[y_n]$, and hence $\overline{Y} \subseteq V$.

(OR: $X - \overline{Y} = \bigcup_{\substack{U\subseteq X-Y\\ U \text{ open}}} U \Rightarrow \overline{Y} = \bigcap_{\substack{U\subseteq X-Y\\ U \text{ open}}} (X-U)$. However, $U \subseteq X-Y$ is open iff $(X-U) \supseteq Y$ is closed.)