Lemma If X is totally bounded, then any sequence $[x_n]$ has a Cauchy subsequence.

Pf: Since X is totally bounded, any subspace is. Thus for any $\epsilon>0$, $\exists N_\epsilon$ s.t. $\{m \mid m\geq N_\epsilon,\ x_m\in B_\epsilon(x_{N_\epsilon})\}$ is infinite: otherwise we need only many $B_\epsilon(x_i)$ to cover $\{x_1,\dots\}\subseteq X$. Now we start describing a subsequence by iteratively building subsequences.

Let $x_{0,n}=x_n$. Assume we have built subsequences $[x_{k,n}]$ for $0\leq k\leq K-1$, and build $x_{K,n}$: Choose N_K s.t. $\{m \mid m\geq N_K,\ x_{K-1,m}\in B_{1/K}(x_{K-1,N_K})\} := \{m_1=N_K<m_2<\dots\}$ is infinite. Now let $[x_{K,n}]$ be the subsequence of $[x_{K-1,n}]$ defined by $x_{K,i}=x_{K-1,m_i}$. In particular for all m, $d(x_{K,1},x_{K,m})<1/K$. In particular, for all $j\geq K$ and all m,

$d(x_{K,1},x_{j,m})<1/K$. Now let $[y_n]$ be the subsequence $y_n=x_{n,1}$. So for $\epsilon>0$, and $N>0$ st. $\frac{1}{N}<\epsilon$, for all $n,m>N$, $d(y_n,y_m)=d(x_{n,1},x_{m,1})<\frac{1}{N}<\epsilon$. □

Cor: (iii) $\Rightarrow$ (ii).

Def A collection of open sets is a basis for the topology on X if every open set U is a union of open sets in the basis.

Prop $\mathcal{B}=\{V_i\}_{i\in I}$ is a basis for the topology on X iff $\forall\, U\subseteq X$ open & $x\in U$, $\exists V\in\mathcal{B}$ s.t. $x\in V\subseteq U$.

Pf: $\Rightarrow$) If any U is a union of basis elements, then any x is visibly in a V in U.

$\Leftarrow$) For each $x\in U$, choose a $V_x\in\mathcal{B}$ w/ $x\in V_x\subseteq U$. Then $U\subseteq\bigcup_{x\in U}V_x\subseteq U$. □

Def X is separable if it has a countable dense subset.

Thm X is separable iff X has a countable basis.

Pf $\Leftarrow$) For each $V\in\mathcal{B}$, choose an element $x_V\in V$. Let $Y=\{x_V \mid V\in\mathcal{B}\}$. Then if U is open, then $\left(U=\bigcup_{\substack{V\subseteq U\\ V\in\mathcal{B}}}V\right)\cap Y=\bigcup_{\substack{V\subseteq U\\ V\in\mathcal{B}}}V\cap Y\neq\emptyset$. So Y is countable and dense.

$\Rightarrow$) Let Y be a countable dense subset. Let $\mathcal{B}=\{B_{1/n}(y) \mid y\in Y,\ n\in\mathbb{N}_{>0}\}$. This is countable. Now let U be open & $x\in U$. Since U is open, for some $N>0$, $B_{1/N}(x)\subseteq U$. Now $B_{1/2N}(x)$ is open, hence $\exists\, y\in Y\cap B_{1/2N}(x)$. Then $x\in B_{1/2N}(y)$ & if $z\in B_{1/2N}(y)$, then

$d(z,x) \le d(z,y) + d(y,x) < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}$. $\Rightarrow x \in \underset{\mathcal{B}}{\underset{\cap}{B_{1/2N}(y)}} \subseteq B_{1/N}(x) \subseteq U$ & $\mathcal{B}$ is a basis. $\square$

<u>Def</u> We say X is <u>second countable</u> if it is separable.

<u>Prop</u> If X is totally bounded, then X is 2^{nd} countable.

<u>Pf</u>: For each $n \in \mathbb{N}_{>0}$, choose elements $x_{n,i}$ $1 \le i \le k_n$ s.t. $X = \bigcup_{i=1}^{k_n} B_{1/n}(x_{n,i})$. We can do this because X is totally bounded. Let $Y = \{x_{n,i} \mid n \in \mathbb{N}_{>0}, 1 \le i \le k_n\}$. Then if $x \in X$, and if $\epsilon > 0$, then for $N \in \mathbb{N}$ s.t. $1/N < \epsilon$, $\exists\, i \le k_N$ s.t. $d(x, x_{N,i}) < \frac{1}{N} < \epsilon$. $\Rightarrow Y$ is dense. $\square$