We have several important, useful equivalent formulations. This will take a while, so state now and get some conclusions.

<u>Def</u> A subspace Y of a metric space X is <u>totally bounded</u> if $\forall \epsilon > 0$, $\exists y_1, \dots, y_n$ s.t. $Y \subseteq B_\epsilon(y_1) \cup \dots \cup B_\epsilon(y_n)$.

A subspace Y is <u>bounded</u> if $\exists r > 0$ s.t. $\forall x, y \in Y$ $d(x,y) < r$.

"Bounded" is <u>not</u> topological: given any X, let $d'(x,y) = \min\{1, d(x,y)\}$. This is a metric that is equivalent to d, and every $Y \subseteq X$ is bounded.

<u>Prop</u> If Y is totally bounded, then Y is bounded.

<u>Pf</u>: Let $\epsilon > 0$, and choose $y_1, \dots, y_n$ s.t. $Y \subseteq B_\epsilon(y_1) \cup \dots B_\epsilon(y_n)$. Then $\{d(y_i, y_j) \mid i \neq j\}$ is a finite set of positive real numbers. Let $r' = \max\{d(y_i, y_j) \mid i \neq j\}$, and $r = 2\epsilon + r'$. Given $y, y' \in Y$, $\exists$ i, j s.t. $y \in B_\epsilon(y_i)$, $y' \in B_\epsilon(y_j)$. $\Rightarrow$ $d(y,y') \leq d(y, y_i) + d(y_i, y') \leq d(y, y_i) + d(y_i, y_j) + d(y_j, y') < 2\epsilon + r' = r$. □

<u>Prop</u> $Y \subseteq \mathbb{R}^n$ is totally bounded iff Y is bounded.

<u>Pf</u>: $\Rightarrow$) is above. If Y is bounded, then for some r, $Y \subseteq [-r, r] \times \dots \times [-r, r]$. This is visibly totally bounded: take some $1/m < \epsilon$ and take balls of radius ϵ at the points $(-r + \frac{k_1}{m}, \dots, -r + \frac{k_n}{m})$, k_i integers s.t. $0 \leq k_i \leq 2mr$. □

<u>Thm</u> If X is a metric space, then the following are equivalent

(i) X is compact

(ii) Any sequence $[x_n]$ in X has a convergent subsequence $[x_{n_k}]$ ("X is <u>sequentially compact</u>")

(iii) X is complete and totally bounded.

<u>Cor</u> (Heine-Borel Thm) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

<u>Pf</u>: Let $X \subseteq \mathbb{R}^n$. X is closed iff X is complete ($\mathbb{R}^n$ is complete). X is bounded iff it is totally bounded □

<u>Lemma</u> (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

<u>Pf</u> (i) $\Rightarrow$ (ii): Let $[x_n]$ be a sequence in X, X compact. If $[x_{n_k}]$ is a convergent

subsequence, then $\exists x \in X$ s.t $\forall \epsilon > 0$, $B_\epsilon(x)$ contains ∞ly many terms in $[x_n]$. Conversely, if $\exists$ $x \in X$ s.t. $\forall \epsilon > 0$, $B_\epsilon(x)$ contains ∞ly many terms, then for each $k > 0$, let x_{n_k} be any element in $B_\epsilon(x)$ with $n_k > n_{k-1}$ (which happens since there are ∞ly many terms). So by construction, $[x_{n_k}]$ is a subsequence converging to x. If $\forall x \in X$, $\exists r_x > 0$ s.t. $B_{r_x}(x)$ contains only finitely many terms, then $X \subseteq \bigcup_{x \in X} B_{r_x}(x)$. X is compact $\Rightarrow \exists y_1, \dots, y_N$ s.t. $X = B_{r_{y_1}}(y_1) \cup \dots \cup B_{r_{y_n}}(y_n)$ & only finitely many of the x_n are in each $B_{r_{y_1}}(y_1), \dots, B_{r_{y_n}}(y_n)$. $\Rightarrow$ ∞ly many terms are missing! So $\exists x \in X$ s.t. $\forall r > 0$, $B_r(x)$ contains ∞ly many terms as desired.

(ii) $\Rightarrow$ (iii) If $[x_n]$ is a Cauchy sequence in X, and $[x_{n_k}]$ is a convergent subsequence, then $[x_n]$ also converges to x, so Cauchy sequences converge & X is complete. To show X is totally bounded, build a sequence of elements directly: Choose $x_1 \in X$ arbitrarily, and let for $n \geq 1$ $x_n \in X - \left(\bigcup_{i=1}^{n-1} B_\epsilon(x_i) \right)$. These have the property that $d(x_i, x_j) \geq \epsilon$ for $1 \leq i \neq j \leq n$. If we can find ∞ly many x_n, then we have a sequence $[x_n]$ w/ $d(x_i, x_j) \geq \epsilon$ $\forall i, j \geq 1$. In particular, we have no <u>Cauchy</u> subsequence of $[x_n]$ & hence no convergent subsequence. $\square$