Let $X_1, \dots, X_n$ be a collection of metric spaces. We want a way to put a metric on

$$X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i\}.$$

There is no unique choice, but there is a "right" notion of open sets: the ones such that the projections are continuous. There is a relative way to say this, which helps in generalizing.

Prop If d is any metric on $X = X_1 \times \dots \times X_n$ s.t. the projections $X \xrightarrow{p_i} X_i$ are continuous, then any set of the form $U_1 \times \dots \times U_i \times \dots \times U_n$, $U_i \subseteq X_i$ open, is open & $V_1 \times \dots \times V_n$, $V_i \subseteq X_i$ closed are all closed.

Pf: Let $U_i \subseteq X_i$ be open, and let $\tilde{U}_i = X_1 \times \dots \times U_i \times \dots \times X_n \subseteq X$. Then $\tilde{U}_i = p_i^{-1}(U_i)$ is open iff p_i is continuous, and hence $U_1 \times \dots \times U_n = \bigcap_{i=1}^{n} \tilde{U}_i$ is open if all are. The same argument works for closed sets. $\square$

This proposition puts a lower bound on the number of open sets we want: any metric must at least have these open sets (and their unions).

Def A metric d on $X_1 \times \dots \times X_n$ is a product metric iff a sequence $[\vec{x}_m]$ in $X_1 \times \dots \times X_n$,

$\vec{x}_m = (x_m^{(1)}, \dots, x_m^{(n)})$, converges to $\vec{x} = (x^{(1)}, \dots, x^{(n)})$ iff for $i = 1, \dots, n$, $[x_m^{(i)}]$ converges to $x^{(i)}$.

Prop If d is a product metric on $X = X_1 \times \dots \times X_n$, then a set is open iff it is a union of sets of the form $U_1 \times \dots \times U_n$, $U_i \subseteq X_i$ open.

Pf If d is a product metric, then by assumption, if $\vec{x}_m \to \vec{x}$, then $p_i(\vec{x}_m) \to p_i(\vec{x})$, $1 \le i \le n$. So $p_1, \dots, p_n$ are all continuous, and hence any set that is a union of set of the form $U_1 \times \dots \times U_n$, w/ $U_i \subseteq X_i$ open, is open.

For the other direction, it suffices to show that if U is open & $\vec{x} \in U$, then $\exists\, U_1 \times \dots \times U_n \subseteq U$, containing x & $U_i \subseteq X_i$ open (since then U is the union of all of these). So assume there is an $\vec{x} \in U$ s.t. $\forall\, U_i \subseteq X_i$ open, $\vec{x} \in U_1 \times \dots \times U_n$, $U_1 \times \dots \times U_n \not\subseteq U$. In particular, for each $m > 0$,

$B_{1/m}(x^{(1)}) \times \dots \times B_{1/m}(x^{(n)}) \not\subseteq U$. Let $\vec{x}_m \in B_{1/m}(x^{(1)}) \times \dots \times B_{1/m}(x^{(n)}) - U$. Then $[\vec{x}_m]$ is a sequence in X that does not converge to $\vec{x}$, since U is an open set about $\vec{x}$ with no $\vec{x}_m$ in it. However, for $1 \le i \le n$, $d(x_m^{(i)}, x^{(i)}) < 1/m$, so $x_m^{(i)} \longrightarrow x^{(i)}$. This contradicts our assumption on the metric on X. $\square$

Thm If $X = X_1 \times \dots \times X_n$ is equipped with a product metric, and if Y is any metric space, then $Y \xrightarrow{f} X$ is continuous iff $\forall\, 1 \le i \le n$, $Y \xrightarrow{f} X \xrightarrow{p_i} X_i$ is continuous.

Pf: Since $p_1, \dots, p_n$ are all continuous for X equipped with product metric, if f is continuous, then $p_1 \circ f, \dots, p_n \circ f$ are all continuous. For the other direction, assume $p_i \circ f$ is continuous for $1 \le i \le n$. Then if $U_i \subseteq X_i$ is open, then $(p_i \circ f)^{-1}(U_i) = f^{-1}(p_i^{-1}(U_i)) = f^{-1}(\underbrace{X_1 \times \dots \times U_i \times \dots \times X_n}_{\widetilde{U}_i})$ is open in Y.

$\Rightarrow f^{-1}(U_1 \times \dots \times U_n) = f^{-1}\left(\bigcap_{i=1}^n \widetilde{U}_i\right) = \bigcap_{i=1}^n f^{-1}(\widetilde{U}_i)$ is a finite intersection of opens $\Rightarrow$ open. Since any $U \subseteq X$ open is of the form $U = \bigcup_{i \in I} \hat{U}_i$, $\hat{U}_i = U_1^i \times \dots \times U_n^i$, where $U_j^i \subseteq X_j$ is open $1 \le j \le n$, $f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} \hat{U}_i\right) = \bigcup_{i \in I} f^{-1}(\hat{U}_i)$ is a union of opens, and hence is open. $\Rightarrow f$ is continuous.

Now any map of sets $Y \xrightarrow{f} X_1 \times \dots \times X_n$ is uniquely determined by the projections:

$f(y) = ((p_1 \circ f)(y), \dots, (p_n \circ f)(y))$, and we can arbitrarily choose the functions $p_i \circ f$: given $f_i : Y \longrightarrow X_i$, $\exists!\ f : Y \longrightarrow X_1 \times \dots \times X_n$ w/ $f(y) = (f_1(y), \dots, f_n(y))$.

Def Let $Map(Y, X)$ be the set of continuous maps $Y \longrightarrow X$.

Cor If $X = X_1 \times \dots \times X_n$ with a product metric, then

$$Map(Y,X) \longrightarrow Map(Y,X_1) \times \dots \times Map(Y,X_n) \longrightarrow Map(Y,X)$$

$$f \longmapsto (p_1 \circ f, \dots, p_n \circ f)$$

$$(f_1, \dots, f_n) \longmapsto (y \longmapsto (f_1(y), \dots, f_n(y)))$$

are inverse functions.

This is the first example of a universal property: everything about $X_1 \times \dots \times X_n$ is recorded in the above corollary. We will run into this many more times!