We are going to start teasing apart the properties of $\mathbb{R}^n$.

<u>Def</u> A <u>metric</u> on a set X is a function $d: X \times X \longrightarrow \mathbb{R}$ s.t.

1) $d(x,y) \geq 0 \quad \forall x,y$ 2) $d(x,y) = 0$ iff $x = y$

3) $d(x,y) = d(y,x) \quad \forall x,y$ 4) $\forall x,y,z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$

<u>Examples</u>: 1) $X = \mathbb{R}$, $d_2(x,y) = |x-y|$ or more generally

1') $X = \mathbb{R}^n$, $d_2(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$. (4) is the "triangle inequality")

2) $X = \mathbb{Q}$, $d(a,b) = p^{-s}$, where $(a-b) = p^s \cdot \frac{c}{d}$, c,d,p relatively prime.

3) $X = \mathbb{R}^n$, $d_\infty(\vec{x},\vec{y}) = \max \{|x_1-y_1|, \dots, |x_n-y_n|\}$

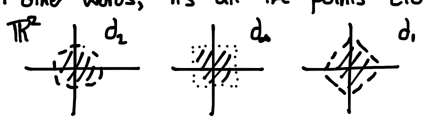
4) $X = \mathbb{R}^n$, $d_p(\vec{x},\vec{y}) = \left(|x_1-y_1|^p + \dots + |x_n-y_n|^p\right)^{1/p}$.

d is a distance function, so it lets us measure closeness and talk about convergence

<u>Def</u> If $x \in X$, $r \in \mathbb{R}_{>0}$, then the <u>open ball of radius r centred at x</u>, $B_r(x)$ is

$$B_r(x) = \{y \in X \mid d(x,y) < r\}.$$

In words, the open ball of radius r is the collection of all points of distance less than r from x. In other words, it's all the points close to x, where we measure "close" by "distance $< r$."

<u>Ex</u>: 

<u>Def</u> A point $y \in Y$, $Y \subseteq X$ is an <u>interior point</u> of Y if $\exists\, r_y \in \mathbb{R}_{>0}$ s.t. $B_{r_y}(y) \subseteq Y$.

In other words, $y \in Y$ is an interior point iff all of the points sufficiently close to y are also in Y.

<u>Prop</u> If $x \in X$, $r \in \mathbb{R}_{>0}$, then every $y \in B_r(x)$ is an interior point of $B_r(x)$.

<u>Pf</u> Let $s = r - d(x,y) > 0$, since $y \in B_r(x)$ iff $d(x,y) < r$. If $z \in B_s(y)$, then we can bound

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + s = r. \quad \text{So } B_s(y) \subseteq B_r(x). \qquad \square$$

<u>Def</u>: The <u>interior</u> of Y, $\text{int}(Y) = \mathring{Y}$, is $\text{int}(Y) = \{y \in Y \mid y \text{ is an interior point of } Y\}$.

A set $U \subseteq X$ is open if $U = \text{int}(U)$.

<u>Example</u>: The open balls are open.

<u>Example</u> If $X \subseteq \mathbb{R}^2$ is carved out by finitely many inequalities (like "$x \leq y$", etc),

then the interior of X is carved out by the strict inequalities (as "$x<y$").

Prop Let U_i for $i \in I$ be a collection of open sets in X, then $U = \bigcup_{i\in I} U_i$ is also open.

Pf: Let $y \in U$. We must show it is an interior point. Since $y \in U$, there is an $i \in I$ s.t. $y \in U_i$.
By assumption, every $y \in U_i$ is an interior point, so there is an r_y s.t. $B_{r_y}(y) \subseteq U_i$. Since $U_i \subseteq U$, we have $B_{r_y}(y) \subseteq U$ $\therefore$ y is an interior point $\square$

Prop If $U_1, \dots, U_n \subseteq X$ are open, then $U_1 \cap \dots \cap U_n$ is open.

Pf Let $U = U_1 \cap \dots \cap U_n$. If $y \in U$, then $y \in U_i$ for $1 \le i \le n$. For each i, U_i is open, so there is an $r_i > 0$ s.t. $B_{r_i}(y) \subseteq U_i$. Since if $r \le r'$, then $B_r(y) \subseteq B_{r'}(y)$, we know that if $r = \min\{r_1, \dots, r_n\}$, then $B_r(y) \subseteq B_{r_i}(y)$ $\forall\, 1 \le i \le n$. In particular, $B_r(y) \subseteq U$, and $y \in \mathring{U}$. $\square$

Remark Infinite intersections need not be open. Where did we use finiteness above? With "min". For an infinite intersection, have to use inf, and while "min" of positive things is positive, "inf" need not be.

Prop If $Y \subseteq X$, then $\mathring{Y} = \bigcup_{\substack{U \subseteq Y \\ \text{open}}} U$.

Pf: If $U \subseteq Y$ is an open set, then every point of U is an interior point of $Y \Rightarrow U \subseteq \mathring{Y}$. So then $\bigcup_{\substack{U \subseteq Y \\ U \text{ open}}} U \subseteq \mathring{Y}$. For the other direction, $\mathring{Y}$ is open, so $\mathring{Y} \subseteq \bigcup_{\substack{U \subseteq Y \\ U \text{ open}}} U$. $\square$