Quiz #7

Solutions

1. Show that the linear operator $D^2 - 4D + 4I$ is 1-1 on $P_2(x)$.

$1\text{-}1 \leftrightarrow \ker = \{0\}$. $\quad \ker = \{ax^2+bx+c \mid (D^2-4D+4I)(ax^2+bx+c) = 0\}$

$$\Rightarrow \quad 4ax^2 + (4b-8a)x + (4c-4b+2a) = 0$$

$$\Rightarrow \left.\begin{array}{lll} 4a = 0 & \Rightarrow & a = 0 \\ 4b-8a = 0 & \Rightarrow & b = 0 \\ 4c-4b+2a = 0 & \Rightarrow & c = 0. \end{array}\right\} \Rightarrow \ker = \{0\}.$$

2. Let $\mathcal{B} = \{1, x+1, x^2+1\}$ be a basis for $P_2(x)$.

   (a) Find the change of basis matrix from $\mathcal{B}$ to the standard basis $\{1, x, x^2\}$.

$$_{\mathcal{E}}C_{\mathcal{B}} = \begin{array}{c c} & \begin{array}{ccc} 1 & x+1 & x^2+1 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

   (b) If $p(x) = 4 + 2x + x^2$, find $[p]_{\mathcal{B}}$.

$$4 + 2x + x^2 = (x^2+1) + 2(x+1) + 1$$

$$\Rightarrow \quad [p]_{\mathcal{B}} = \begin{array}{c c} \begin{array}{c} 1 \\ x+1 \\ x^2+1 \end{array} & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{array}$$

3. Let $L(p) = p'' - 4p' + 4p$, just as above, and let $\mathcal{E}$ be the standard basis for $P_2(x)$. Find the matrix of L with respect to the basis $\mathcal{E}$: $_{\mathcal{E}}[L]_{\mathcal{E}}$.

$L(1) = 4$

$L(x) = 4x - 4$

$L(x^2) = 4x^2 - 8x + 2$

$$\Rightarrow \quad _{\mathcal{E}}[L]_{\mathcal{E}} = \begin{array}{c c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \end{array} & \begin{bmatrix} 4 & -4 & 2 \\ 0 & 4 & -8 \\ 0 & 0 & 4 \end{bmatrix} \end{array}$$