Quiz # 6

Solutions

April 3, 2008

1. Find a basis for the kernel and range of the linear transformation $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by
$$L(\mathbf{v}) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \mathbf{v}.$$

kernel: $\{\bar{v} \mid L(\bar{v}) = \bar{0}\} = \{\bar{v} \mid \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \bar{v} = \bar{0}\}$ Row reduce: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ x, z lead, y free $\Rightarrow$ sol: $\left\{ \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} \right\}$, so a basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Range: Row reduce $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow$ basis vectors

$\Rightarrow$ basis is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

(Note that another basis is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$, the columns corresponding to the leading rows in the RRE form for the kernel)

2. Use Gram-Schmidt to turn $\{(1,1,3),(0,3,1),(1,1,0)\}$ into an orthogonal basis (it doesn't have to be orthonormal).

$\bar{w}_1 = (1,1,3)$ $\bar{w}_1 \cdot \bar{w}_1 = 11$, $\bar{w}_1 \cdot \bar{v}_2 = 6$, $\bar{w}_1 \cdot \bar{v}_3 = 2$

$$\bar{w}_2 = \bar{v}_2 - \frac{\bar{w}_1 \cdot \bar{v}_2}{\bar{w}_1 \cdot \bar{w}_1} \bar{w}_1 = (0,3,1) - \frac{6}{11}(1,1,3) = \left(-\frac{6}{11}, \frac{27}{11}, \frac{-7}{11}\right)$$

$$\bar{w}_2 \cdot \bar{w}_2 = \frac{36}{121} + \frac{729}{121} + \frac{49}{121} = \frac{814}{121} = \frac{74}{11}$$
$$\bar{w}_2 \cdot \bar{v}_3 = \frac{21}{11}$$

$$\bar{w}_3 = \bar{v}_3 - \frac{\bar{w}_1 \cdot \bar{v}_3}{\bar{w}_1 \cdot \bar{w}_1} \bar{w}_1 - \frac{\bar{w}_2 \cdot \bar{v}_3}{\bar{w}_2 \cdot \bar{w}_2} \bar{w}_2$$

(This is much harder than you'll have to do. You can make things easier by doing a different order)

$$= (1,1,0) - \frac{2}{11}(1,1,3) - \frac{21/11}{74/11}\left(-\frac{6}{11}, \frac{27}{11}, \frac{-7}{11}\right)$$

$$= \left(\frac{9}{11}, \frac{9}{11}, \frac{-6}{11}\right) - \frac{21}{74}\left(\frac{-6}{11}, \frac{27}{11}, \frac{-7}{11}\right)$$

$$= \frac{1}{11}\left(\frac{792}{74}, \frac{99}{74}, \frac{-297}{74}\right)$$