Name: Solutions ☐ TTH ☐ MWF 10AM ☐ MWF 11AM

Pledge: ______________________________

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Signature: ______________________________

To get credit for a problem, you must show all of your reasoning and calculations. No calculators may be used.

Problem	**Score**
1	
2	
3	
4	
5	
Total	

1. (a) (12 Points) Find the adjoint, determinant and inverse of

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

(b) (8 Points) Compute the determinant of B using elimination method

$$B = \begin{bmatrix} 2 & 3 & 0 & 2 \\ 2 & 6 & -1 & 5 \\ 2 & 3 & -1 & -4 \\ 4 & 12 & -3 & 7 \end{bmatrix}.$$

1a) $\left|\begin{matrix} 1 & 3 & 4 \\ -4 & 2 & 1 \\ 2 & 1 & 2 \end{matrix}\right|\begin{matrix} 1 & 3 \\ -4 & 2 \\ 2 & 1 \end{matrix} = (4+6-16)-(16+1-24)$

$= \underline{1}$

Matrix of cofactors:

$C_{1,1} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$C_{1,2} = (-1)^{1+2}\begin{vmatrix} -4 & 1 \\ 2 & 2 \end{vmatrix} = 10$

$C_{1,3} = (-1)^{1+3}\begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} = -8$

$C_{2,1} = (-1)^{2+1}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2$

$C_{2,2} = (-1)^{2+2}\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} = -6$

$C_{2,3} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$

$C_{3,1} = (-1)^{3+1}\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$

$C_{3,2} = (-1)^{3+2}\begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} = -17$

$C_{3,3} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 14$

$\Rightarrow C = \begin{bmatrix} 3 & 10 & -8 \\ -2 & -6 & 5 \\ -5 & -17 & 14 \end{bmatrix}$

Adjoint: $\operatorname{adj}(A) = C^t$

$= \begin{bmatrix} 3 & -2 & -5 \\ 10 & -6 & -17 \\ -8 & 5 & 14 \end{bmatrix}$

$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \operatorname{adj}(A) =$

$\begin{bmatrix} 3 & -2 & -5 \\ 10 & -6 & -17 \\ -8 & 5 & 14 \end{bmatrix}$

1b)

$$\begin{vmatrix} 2 & 3 & 0 & 2 \\ 2 & 6 & -1 & 5 \\ 2 & 3 & -1 & -4 \\ 4 & 12 & -3 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 6 & -3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 2 \cdot 3 \cdot (-1) \cdot 3 = -18$$

2. (a) (10 Points) Check whether the vectors

$$w_1 = (-1, -16, -7) \text{ and } w_2 = (1, 2, 3)$$

lie in the space generated by the vectors

$$v_1 = (1, -2, 1),\ v_2 = (-2, 1, -3) \text{ and } v_3 = (-1, -4, -3).$$

If w_1 or w_2 lie in the span$\{v_1, v_2, v_3\}$ write it as a linear combination of v_1, v_2, v_3 (Hint: Solve the two resulting systems simultaneously).

(b) (10 points) Check if the following set of matrices linearly independent:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 6 \end{bmatrix}.$$

2a) $a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 = \begin{bmatrix} a - 2b - c \\ -2a + b - 4c \\ a - 3b - 3c \end{bmatrix}$ so w_1 / w_2 in span iff

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -4 \\ 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = w_1 / w_2 \quad \text{has solutions:}$$

$$\begin{array}{c} \\ -1 \end{array} \left[\begin{array}{ccc|cc} & & & w_1 & w_2 \\ 1 & -2 & -1 & -1 & 1 \\ -2 & 1 & -4 & -16 & 2 \\ 1 & -3 & -3 & -7 & 3 \end{array}\right] 2 \Rightarrow \left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ 0 & -3 & -6 & -18 & 4 \\ 0 & -1 & -2 & -6 & 2 \end{array}\right] -1$$

$$\Rightarrow \left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ 0 & 1 & 2 & 6 & 2 \\ 0 & -3 & -6 & -18 & 4 \end{array}\right] 3 \Rightarrow \left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ 0 & 1 & 2 & 6 & 2 \\ 0 & 0 & 0 & 0 & 10 \end{array}\right]$$

has sol   no solutions

$\Rightarrow$ w_1 in the span, w_2 is not

$$a\begin{bmatrix}1&0\\0&2\end{bmatrix}+b\begin{bmatrix}0&1\\-1&0\end{bmatrix}+c\begin{bmatrix}-1&0\\0&2\end{bmatrix}+d\begin{bmatrix}1&2\\-2&6\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\updownarrow$$

$$\begin{aligned} a-c+d &= 0\\ b+2d &= 0\\ -b-2d &= 0\\ 2a+2c+6d &= 0\end{aligned} \quad\longleftrightarrow\quad \begin{bmatrix}1&0&-1&1\\0&1&0&2\\0&-1&0&-2\\2&0&2&6\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Row Reduce:

$$-2\left(\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&2&0\\0&-1&0&-2&0\\2&0&2&6&0\end{array}\right)+ \;\Rightarrow\; \left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&2&0\\0&0&0&0&0\\0&0&4&4&0\end{array}\right]\div 4$$

$$\Rightarrow \left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&2&0\\0&0&1&1&0\\0&0&0&0&0\end{array}\right]$$

has non-trivial solutions (fewer non-zero rows than # of columns)

$\Rightarrow$ lin dependent $\qquad \left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}-2\\-2\\-1\\1\end{bmatrix}\text{ works}\right)$

3. (a) (20 Points) Determine a basis for the kernel and range of the transformation defined by the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -2 \\ 2 & 20 & 22 \end{bmatrix},$$

and verify the Rank-Nullity theorem.

Kernel:

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -2 \\ 2 & 20 & 22 \end{bmatrix} \Rightarrow \div 14 \begin{bmatrix} 1 & 3 & 4 \\ 0 & 14 & 14 \\ 0 & 14 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ x,y lead, z free : Sol to $A\bar{v} = \bar{0} \longleftrightarrow \bar{v} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix}$, so

$$\text{basis} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Range:

$$A^t = \begin{bmatrix} 1 & -4 & 2 \\ 3 & 2 & 20 \\ 4 & -2 & 22 \end{bmatrix} \Rightarrow \div 14 \begin{bmatrix} 1 & -4 & 2 \\ 0 & 14 & 14 \\ 0 & 14 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{basis vect.}$$

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\underbrace{\dim \ker}_{\text{Nullity}} = 1$, $\underbrace{\dim \text{ range}}_{\text{rank}} = 2$, and rank + nullity = 3 = # col. ✓

4. (20 Points)Find eigenvalues and and the set of all eigenvectors for each eigenvalue of the following matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$p_A(\lambda) = |A-\lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = \Big((2-\lambda)\underbrace{(2-\lambda)(-2-\lambda)}^{\lambda^2-4} - 3\cdot1\cdot1 - 3\cdot1\cdot1\Big) - \big(1\cdot(-3)\cdot(2-\lambda) + 1\cdot(-2-\lambda)\cdot1 + 3\cdot1\cdot(2-\lambda)\big)$$

$$= (2\lambda^2 - \lambda^3 + 4\lambda - 14) - (3\lambda - 6 - \lambda - 2 + 3\lambda - 6)$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2$$

So $p_A(\lambda) = 0 \longleftrightarrow \lambda = 0, \ \lambda = 1$

$\lambda = 0$: $(A-\lambda I)\bar{v} = \bar{0} \Rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}\bar{v} = \bar{0}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $(A-\lambda I)\bar{v} = \bar{0} \longleftrightarrow \bar{v} = \begin{bmatrix} r \\ r \\ r \end{bmatrix}$, so eignvectors are

$$\left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix}, r \neq 0 \right\}$$

$\lambda = 1$: $(A-\lambda I)\bar{v} = \bar{0}$: $\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}\bar{v} = \bar{0}$

$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$(A-I)\bar{v} = \bar{0} \longleftrightarrow \bar{v} = \begin{bmatrix} 3r-s \\ r \\ s \end{bmatrix}$, so

eignvectors: $\left\{ \begin{bmatrix} 3r-s \\ r \\ s \end{bmatrix}, \text{ r,s not both 0} \right\}$

5. (20 Points) Use Gram-Schmidt process to obtain an orthogonal basis for $\mathbb{R}^3$ from the basis $\{v_1 = (1,1,1),\ v_2 = (-1,0,1),\ v_3 = (-1,2,3)\}$.

$\bar{w}_1 = \bar{v}_1 = (1,1,1)$ $\qquad \bar{w}_1 \cdot \bar{v}_2 = 0$ $\qquad \bar{w}_1 \cdot \bar{v}_3 = 4$

$\bar{w}_1 \cdot \bar{w}_1 = 3$

$$\bar{w}_2 = \bar{v}_2 - \mathrm{Proj}_{\bar{w}_1} \bar{v}_2 = \bar{v}_2 - \frac{\bar{v}_2 \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1} \bar{w}_1 = \bar{v}_2 = (-1,0,1)$$

$\bar{w}_2 \cdot \bar{w}_2 = 2$

$\bar{w}_2 \cdot \bar{v}_3 = 4$

$$\bar{w}_3 = \bar{v}_3 - \mathrm{Proj}_{\bar{w}_1} \bar{v}_3 - \mathrm{Proj}_{\bar{w}_2} \bar{v}_3 = \bar{v}_3 - \frac{\bar{v}_3 \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1} \cdot \bar{w}_1 - \frac{\bar{v}_3 \cdot \bar{w}_2}{\bar{w}_2 \cdot \bar{w}_2} \bar{w}_2$$

$$= (-1,2,3) - \frac{4}{3}(1,1,1) - \frac{4}{2}(-1,0,1)$$

$$= \left(\frac{-7}{3}, \frac{2}{3}, \frac{5}{3}\right) + (2,0,-2)$$

$$= \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$