APMA 308 MIDTERM #2 APRIL 8TH, 2008

NAME: Solutions ☐ TTH ☐ MWF 10AM ☐ MWF 11AM

PLEDGE: ______________________________

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SIGNATURE: ______________________________

To get credit for a problem, you must show all of your reasoning and calculations. No calculators may be used.

London

Qian

Paramasamy

Mike

Sabrina

Problem	Score
1	
2	
3	
4	
5	
Total	

1. (a) (12 Points) Find the adjoint, determinant and inverse of

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

(b) (8 Points) Compute the determinant of B using elimination method

$$B = \begin{bmatrix} 2 & 3 & 0 & 2 \\ 2 & 6 & -1 & 5 \\ 2 & 3 & -1 & -4 \\ 4 & 12 & -3 & 7 \end{bmatrix}.$$

1a) $\left|\begin{matrix} 1 & 3 & 4 \\ -4 & 2 & 1 \\ 2 & 1 & 2 \end{matrix}\right| \begin{matrix} 1 & 3 \\ -4 & 2 \\ 2 & 1 \end{matrix} = (4+6-16)-(16+1-24) = \underline{1}$ } 3 points

Matrix of cofactors:

$C_{1,1} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$C_{1,2} = (-1)^{1+2}\begin{vmatrix} -4 & 1 \\ 2 & 2 \end{vmatrix} = 10$

$C_{1,3} = (-1)^{1+3}\begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} = -8$

$C_{2,1} = (-1)^{2+1}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2$

$C_{2,2} = (-1)^{2+2}\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} = -6$

$C_{2,3} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$

-2 for forgetting signs

6 points

$C_{3,1} = (-1)^{3+1}\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$

$C_{3,2} = (-1)^{3+2}\begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} = -17$

$C_{3,3} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 14$

$\Rightarrow C = \begin{bmatrix} 3 & 10 & -8 \\ -2 & -6 & 5 \\ -5 & -17 & 14 \end{bmatrix}$

Adjoint: $\text{adj}(A) = C^t = \begin{bmatrix} 3 & -2 & -5 \\ 10 & -6 & -17 \\ -8 & 5 & 14 \end{bmatrix}$

$A^{-1} = \frac{1}{\det A}\,\text{adj}(A) = \text{adj}(A) = \begin{bmatrix} 3 & -2 & -5 \\ 10 & -6 & -17 \\ -8 & 5 & 14 \end{bmatrix}$ } 3 pts

1b)

$$\underbrace{-\begin{vmatrix} 2 & 3 & 0 & 2 \\ 2 & 6 & -1 & 5 \\ 2 & 3 & -1 & -4 \\ 4 & 12 & -3 & 7 \end{vmatrix} \begin{matrix} - \\ -2 \\ \\ \\ \end{matrix} = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 6 & -3 & 3 \end{vmatrix} \begin{matrix} \\ -2 \\ \\ \\ \end{matrix} = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & -1 & -3 \end{vmatrix} \begin{matrix} \\ \\ -1 \\ \\ \end{matrix} = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 3 \end{vmatrix}}_{\text{+6 through here (-1 each false det rule)}} = \underbrace{2 \cdot 3 \cdot (-1) \cdot 3 = -18}_{\text{+2 here}}$$

2. (a) (10 Points) Check whether the vectors

$$w_1 = (-1, -16, -7) \text{ and } w_2 = (1, 2, 3)$$

lie in the space generated by the vectors

$$v_1 = (1, -2, 1),\ v_2 = (-2, 1, -3) \text{ and } v_3 = (-1, -4, -3).$$

If w_1 or w_2 lie in the span$\{v_1, v_2, v_3\}$ write it as a linear combination of v_1, v_2, v_3 (Hint: Solve the two resulting systems simultaneously).

(b) (10 points) Check if the following set of matrices linearly independent:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 6 \end{bmatrix}.$$

2a) $a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 = \begin{bmatrix} a - 2b - c \\ -2a + b - 4c \\ a - 3b - 3c \end{bmatrix}$ so w_1 / w_2 in span iff $\Big\}+2$

$= \begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -4 \\ 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = w_1 / w_2$ has solutions :

$$\begin{array}{c} \\ \end{array}\left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ -2 & 1 & -4 & -16 & 2 \\ 1 & -3 & -3 & -7 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ 0 & -3 & -6 & -18 & 4 \\ 0 & -1 & -2 & -6 & 2 \end{array}\right]$$

(columns w_1, w_2; row operations: 2, -1, -1)

$$\Rightarrow \left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ 0 & 1 & 2 & 6 & 2 \\ 0 & -3 & -6 & -18 & 4 \end{array}\right] \Big) 3 \Rightarrow \left[\begin{array}{ccc|cc} 1 & -2 & -1 & -1 & 1 \\ 0 & 1 & 2 & 6 & 2 \\ 0 & 0 & 0 & 0 & 10 \end{array}\right] \Big\} +4$$

has sol   no solutions

$\Rightarrow$ w_1 in the span, w_2 is not $\Big\}+2$ (based on what they found))

$$\left.\begin{array}{c} a\begin{bmatrix}1&0\\0&2\end{bmatrix}+b\begin{bmatrix}0&1\\-1&0\end{bmatrix}+c\begin{bmatrix}-1&0\\0&2\end{bmatrix}+d\begin{bmatrix}1&2\\-2&6\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix} \\ \updownarrow \\ \begin{aligned} a-c+d&=0\\ b+2d&=0\\ -b-2d&=0\\ 2a+2c+6d&=0\end{aligned} \longleftrightarrow \begin{bmatrix}1&0&-1&1\\0&1&0&2\\0&-1&0&-2\\2&0&2&6\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\end{array}\right\} +4$$

+4

Row Reduce:

$$\left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&2&0\\0&-1&0&-2&0\\2&0&2&6&0\end{array}\right] \Rightarrow \left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&2&0\\0&0&0&0&0\\0&0&4&4&0\end{array}\right] \div 4$$

$$\Rightarrow \left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&0&2&0\\0&0&1&1&0\\0&0&0&0&0\end{array}\right]$$

has non-trivial solutions (fewer non-zero rows than # of columns)

$\Rightarrow$ $\underbrace{\text{lin dependent}}_{+2}$

$\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}-2\\-2\\-1\\1\end{bmatrix} \text{ works}\right)$

not necessary.

3. (a) (20 Points) Determine a basis for the kernel and range of the transformation defined by the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -2 \\ 2 & 20 & 22 \end{bmatrix},$$

and verify the Rank-Nullity theorem.

Kernel: $\cdot 2 \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -2 \\ 2 & 20 & 22 \end{bmatrix} 4 \Rightarrow \div 14 \begin{bmatrix} 1 & 3 & 4 \\ 0 & 14 & 14 \\ 0 & 14 & 14 \end{bmatrix} - \Rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} -3 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ x, y lead, z free : Sol to $A\bar{v} = \bar{0} \longleftrightarrow \bar{v} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix}$, so

basis $= \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Range: $A^t = -4 \begin{bmatrix} 1 & -4 & 2 \\ 3 & 2 & 20 \\ 4 & -2 & 22 \end{bmatrix} -3 \Rightarrow \div 14 \begin{bmatrix} 1 & -4 & 2 \\ 0 & 14 & 14 \\ 0 & 14 & 14 \end{bmatrix} - \Rightarrow \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} 4 \Rightarrow \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ← basis vect.

basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\underbrace{\text{dim ker}}_{\text{Nullity}} = 1$, $\underbrace{\text{dim range}}_{\text{rank}} = 2$, and rank + nullity $= 3 = \#$ col. ✓

4. (20 Points)Find eigenvalues and and the set of all eigenvectors for each eigenvalue of the following matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$p_A(\lambda) = |A-\lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = \Big((2-\lambda)\overbrace{(2-\lambda)(-2-\lambda)}^{\lambda^2-4} - 3\cdot 1\cdot 1 - 3\cdot 1\cdot 1\Big) - \Big(1\cdot(-3)\cdot(2-\lambda) + 1\cdot(-2-\lambda)\cdot 1 + 3\cdot 1\cdot(2-\lambda)\Big)$$

$$= (2\lambda^2 - \lambda^3 + 4\lambda - 14) - (3\lambda - 6 - \lambda - 2 + 3\lambda - 6)$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2$$

So $p_A(\lambda) = 0 \longleftrightarrow \lambda = 0, \ \lambda = 1$

$\lambda = 0$: $(A-\lambda I)\bar{v} = \bar{0} \Rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}\bar{v} = \bar{0}$

$$-1\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -3 & 2 \end{bmatrix} -2 \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} 2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $(A-\lambda I)\bar{v} = \bar{0} \longleftrightarrow \bar{v} = \begin{bmatrix} r \\ r \\ r \end{bmatrix}$, so eignvectors are

$$\left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix}, \ r \neq 0 \right\}$$

$\lambda = 1$: $(A-\lambda I)\bar{v} = \bar{0}$: $\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}\bar{v} = \bar{0}$

$$\cdot\begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} - \Rightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$(A-I)\bar{v} = \bar{0} \longleftrightarrow \bar{v} = \begin{bmatrix} 3r-s \\ r \\ s \end{bmatrix}$, so

eignvectors: $\left\{ \begin{bmatrix} 3r-s \\ r \\ s \end{bmatrix}, \ r,s \text{ not both } 0 \right\}$

5. (20 Points) Use Gram-Schmidt process to obtain an orthogonal basis for $\mathbb{R}^3$ from the basis $\{v_1 = (1,1,1),\ v_2 = (-1,0,1),\ v_3 = (-1,2,3)\}$.

+2 $\bar{w}_1 = \bar{v}_1 = (1,1,1)$

$\bar{w}_1 \cdot \bar{v}_2 = 0$ $\quad \bar{w}_1 \cdot \bar{v}_3 = 4$

$\bar{w}_1 \cdot \bar{w}_1 = 3$

+4 $\bar{w}_2 = \bar{v}_2 - \text{Proj}_{\bar{w}_1} \bar{v}_2 = \bar{v}_2 - \dfrac{\bar{v}_2 \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1} \bar{w}_1 = \bar{v}_2 = (-1,0,1)$ — +1

+ 8

$\bar{w}_2 \cdot \bar{w}_2 = 2$

$\bar{w}_2 \cdot \bar{v}_3 = 4$

+4

$\bar{w}_3 = \bar{v}_3 - \text{Proj}_{\bar{w}_1} \bar{v}_3 - \text{Proj}_{\bar{w}_2} \bar{v}_3 = \bar{v}_3 - \dfrac{\bar{v}_3 \cdot \bar{w}_1}{\bar{w}_1 \cdot \bar{w}_1} \cdot \bar{w}_1 - \dfrac{\bar{v}_3 \cdot \bar{w}_2}{\bar{w}_2 \cdot \bar{w}_2} \bar{w}_2$

$= (-1,2,3) - \frac{4}{3}(1,1,1) - \frac{4}{2}(-1,0,1)$

$= \left(\frac{-7}{3}, \frac{2}{3}, \frac{5}{3}\right) + (2,0,-2)$

$= \underbrace{\left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)}_{+1}$ (can be scaled, so $(-1,2,-1)$ works too, etc)