APMA 308 | MIDTERM #1 | FEBRUARY 26^{TH}, 2008

NAME: ______________________________ SECTION: ______________

PLEDGE: ____

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SIGNATURE: ______________________________

To get credit for a problem, you must show all of your reasoning and calculations. No calculators may be used.

Problem	Score
1	
2	
3	
4	
5	
Total	

1. (20 pts) Solve the following homogeneous system of linear equations and find a basis for and the dimension of the subspace of the solutions.

$$x_1 + x_2 - 2x_3 + 3x_4 = 0$$
$$2x_1 + x_2 - 5x_3 + 2x_4 = 0$$
$$3x_1 + x_2 - 8x_3 + x_4 = 0$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -5 & 2 & 0 \\ 3 & 1 & -8 & 1 & 0 \end{bmatrix} \approx \begin{array}{c} \\ R_2-2R_1 \\ R_3-3R_1 \end{array} \begin{bmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -1 & -4 & 0 \\ 0 & -2 & -2 & -8 & 0 \end{bmatrix} \approx (-1)R_2 \begin{bmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & -2 & -2 & -8 & 0 \end{bmatrix}$$

$$\approx \begin{array}{c} R_1-R_2 \\ \\ R_3+2R_2 \end{array} \begin{bmatrix} 1 & 0 & -3 & -1 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous system is

$x_3 = r$, $x_4 = s$, $x_2 = -x_3 - 4x_4 = -r - 4s$

$x_1 = 3x_3 + x_4 = 3r + s$

The subspace of solutions $= \left\{ \begin{bmatrix} 3r+s \\ -r-4s \\ r \\ s \end{bmatrix} \quad r, s \in \mathbb{R} \right\}$

Any vector (solution) in this space can be written uniquely as $r \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

for some real numbers r and s (scalars)

Thus $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ form a basis and hence the dimension of the subspace of solutions is 2.

2. (20*pts*) Solve the system of linear equations by determining the inverse of the matrix of coefficients and then using matrix multiplication.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 - 3x_3 &= 0 \\ 2x_1 + 3x_2 - 3x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 5 & -3 & 0 & 1 & 0 \\ 2 & 3 & -3 & 0 & 0 & 1 \end{bmatrix} \approx \begin{matrix} R_2-3R_1 \\ \\ R_3-2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

$$\overset{(-1)R_2}{\approx} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix} \approx_{R_3+R_2} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\approx \begin{matrix} R_1-R_3 \\ R_2-R_3 \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -3 & 5 & -6 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \approx \begin{matrix} R_1-2R_2 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 6 & -9 & 11 \\ 0 & 1 & 0 & -3 & 5 & -6 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

The inverse of the matrix of coefficients is $\begin{bmatrix} 6 & -9 & 11 \\ -3 & 5 & -6 \\ 1 & -1 & 1 \end{bmatrix}$

The solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 & -9 & 11 \\ -3 & 5 & -6 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 28 \\ -15 \\ 3 \end{bmatrix}$

3. (20pts) Solve the following system of linear equations using the method of LU decomposition.

$$2x_1 - x_2 + x_3 = 4$$
$$4x_1 - x_2 + 4x_3 = 0$$
$$2x_1 + x_3 = 2$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \underset{R_3 - R_1}{\overset{R_2 - 2R_1}{\approx}} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \underset{R_3 - R_2}{\approx} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}.$$

The given system of linear equation can be written as $AX = B$.

$LUX = B$. Set $UX = Y$. Then $LY = B$

ie
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$y_1 = 4$, $y_2 = 0 - 2y_1 = -8$

$y_3 = 2 - y_1 - y_2 = 6$

Now solve $UX = Y$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 6 \end{bmatrix}$$

$-2x_3 = 6 \quad x_3 = -3$

$x_2 = -8 - 2x_3 = -2$

$2x_1 = 4 + x_2 - x_3 = 5$

$x_1 = 5/2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -3 \end{bmatrix}$$

4. (a) (12pts) Find a basis for the subspace of all vectors orthogonal to both $(1,2,1,2)$ and $(2,5,3,1)$.

   (b) (8pts) Check whether the vectors $(1,1,1)$, $(1,1,-1)$, and $(-1,1,1)$ are linearly independent or not

Let (x_1, x_2, x_3, x_4) be a vector orthogonal to both $(1,2,1,2)$ and $(2,5,3,1)$. Then by definition

$$x_1 + 2x_2 + x_3 + 2x_4 = 0$$
$$2x_1 + 5x_2 + 3x_3 + x_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 5 & 3 & 1 & 0 \end{bmatrix} \approx_{R_2-2R_1} \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & -3 & 0 \end{bmatrix} \approx_{R_1-2R_2} \begin{bmatrix} 1 & 0 & -1 & 8 & 0 \\ 0 & 1 & 1 & -3 & 0 \end{bmatrix}$$

The general solution: $x_3 = r \quad x_4 = s \quad x_1 = r - 8s$
$x_2 = -r + 3s$

The subspace of orthogonal vectors to the given vectors $= \left\{ \begin{bmatrix} r-8s \\ -r+3s \\ r \\ s \end{bmatrix} \; r, s \in \mathbb{R} \right\}$

A basis for this subspace is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

— x —

$\{(1,1,1), (1,1,-1), (-1,1,1)\}$ is linearly independent iff
$$c_1 + c_2 - c_3 = 0$$
$$c_1 + c_2 + c_3 = 0$$
$$c_1 - c_2 + c_3 = 0$$
has only the zero solution

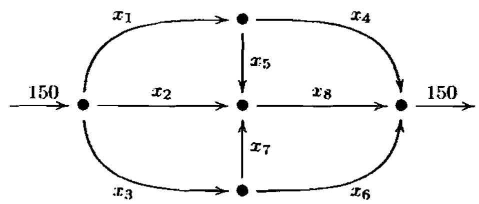
$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \approx_{\substack{R_2-R_1 \\ R_3-R_1}} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{bmatrix} \approx_{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$2c_3 = 0$$
$$-2c_2 + 2c_3 = 0$$
$$c_1 + c_2 - c_3 = 0$$

$\Rightarrow c_3 = 0, \; c_2 = 0, \; c_1 = 0.$

$\therefore$ the given vector are linearly independent

5. (a) (6pts) Construct a system of linear equations that describes the traffic flow in the road network of the following figure. (need not solve).



(b) (6pts) Check whether $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, T(x, y) = (x + y, xy)$ is a linear transformation or not.

(c) (8pts) Show that $W = \{(a, b, c)|a + b + c = 0; a, b, c \in \mathbb{R}\}$ is a subspace.

a)
$$x_1 + x_2 + x_3 = 150$$
$$x_3 - x_6 - x_7 = 0$$
$$x_4 + x_6 + x_8 = 150$$
$$x_1 - x_4 - x_5 = 0$$
$$x_2 + x_5 + x_7 - x_8 = 0$$

— x —

b) Recall T is a linear transformation if
$T(u+v) = T(u) + T(v)$
$T(cu) = cT(u)$ for all vectors $u, v \in \mathbb{R}^2$ and scalars $c \in \mathbb{R}$

Let $u = (x, y)$ and $c \in \mathbb{R}$.
$$T(cu) = T((cx, cy)) = (cx + cy,\ c^2 xy)$$
$$cT(u) = c(x+y, xy) = (cx + cy,\ cxy)$$
If $c \neq 0$, $c \neq 1$ then $T(cu) \neq cT(u)$ $\Rightarrow$ T is not a linear transformation

— x —

c) Recall: W $(\neq \emptyset)$ is a subspace if it is closed under addition and scalar multiplication.

Let $(a_1, b_1, c_1), (a_2, b_2, c_2) \in W \Rightarrow a_1 + b_1 + c_1 = 0 \quad a_2 + b_2 + c_2 = 0$

$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2)$ and

$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = a_1 + b_1 + c_1 + a_2 + b_2 + c_2 = 0 + 0 = 0$

$\Rightarrow (a_1, b_1, c_1) + (a_2, b_2, c_2) \in W$ ie W is closed under addition

Let α be any scalar.

$\alpha(a_1, b_1, c_1) = (\alpha a_1, \alpha b_1, \alpha c_1)$ and

$$\alpha a_1 + \alpha b_1 + \alpha c_1 = \alpha(a_1 + b_1 + c_1)$$
$$= \alpha 0 = 0$$

$\Rightarrow \alpha(a_1, b_1, c_1) \in W$ ie W is closed under Scalar multiplication.

Thus W is a Subspace.