Lecture 8 – Inverse Matrices

Note Title 2/11/2008

Finished last time with 2 matrix operations: Tr & t. Today we will do another big one, closely connected to systems: matrix inverse.

<u>Def</u> An $n\times n$ matrix A is <u>invertible</u> if there is a matrix B, the <u>inverse</u> such that

$$AB = BA = I_n .$$

<u>Ex</u>: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is invertible w/ inverse $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$:

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 6-6 \\ -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Not every matrix has an inverse:

Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

If a matrix A has an inverse, it is unique:

Assume $BA = AB = I = CA = AC \Rightarrow C = C\cdot I_n = C(AB) = (CA)B = I_n \cdot B = B.$

⇝ Makes sense to say B is <u>the</u> inverse to A: $B = A^{-1}$

⇝ Negative powers! If A is invertible, $A^{-k} = (A^{-1})^k$.

Why is this concept important?

If A is invertible, then for <u>any</u> $\bar{b}$, $A\bar{x} = \bar{b}$ has a unique solution:

$\bar{x} = A^{-1}\bar{b}$.

check: $A\cdot(A^{-1}\bar{b}) = (A\cdot A^{-1})\bar{b} = I\cdot\bar{b} = \bar{b}$.

If $\bar{x}$ is a solution, then

$$\begin{array}{ccc} A^{-1}(A\bar{x}) & = & A^{-1}b \\ \parallel & & \parallel \\ (A^{-1}A)\cdot\bar{x} & = & \bar{x} \end{array}$$

⇒ If A is invertible, then the RRE form of A is the identity matrix.

See soon the converse is true.

Ex: $A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$ $\quad \bar{b}=\begin{bmatrix}3\\ 4\end{bmatrix}$

$$A\bar{x}=\bar{b} \Rightarrow \quad \bar{x}=A^{-1}\bar{b}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}3\\ 4\end{bmatrix}=\begin{bmatrix}6-12\\ -3+8\end{bmatrix}=\begin{bmatrix}-6\\ 5\end{bmatrix}.$$

How can we find the inverse?

Solve the systems $\quad A\bar{x}=\bar{e}_1, \dots, \; A\bar{x}=\bar{e}_n$

Ie row reduce $\left[A \,\middle|\, I_n\right] \xrightarrow{GJ} \left[I_n \,\middle|\, A^{-1}\right]$

If A is not invertible, then the RRE form $\neq I_n$.

Ex: $\left[\begin{array}{cc|cc}2 & 3 & 1 & 0\\ 1 & 2 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{cc|cc}1 & 2 & 0 & 1\\ 2 & 3 & 1 & 0\end{array}\right] \xrightarrow{-2} \left[\begin{array}{cc|cc}1 & 2 & 0 & 1\\ 0 & -1 & 1 & -2\end{array}\right] \div -1$

$$\Rightarrow \left[\begin{array}{cc|cc}1 & 0 & 2 & -3\\ 0 & 1 & -1 & 2\end{array}\right] \quad \checkmark$$

Our block method shows this works. If $\bar{b}_i$ is the i^{th} column of A^{-1}, then

$$AA^{-1}=\left[A\bar{b}_1 \middle| \cdots \middle| A\bar{b}_n\right]=\left[\bar{e}_1 \middle| \cdots \middle| \bar{e}_n\right]=I_n \iff A\bar{b}_i=\bar{e}_i$$

Nicer, very interesting story:

Def An *elementary* matrix is the result of applying an elementary operation to I_n.

Ex:

- $\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ (swapped rows 1 & 3)

- $\begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ (scale row 1 by 3)

- $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$ (add 4 times row 3 to row 2)

→ Row operations correspond to elementary matrices.

If A is an $n \times m$ matrix, then performing a row operation to A is the same as left multiplying by the corresponding elementary matrix.

Since row ops are invertible, so are elementary matrices

So Gauss-Jordan is a series of matrix mults, one for each row operation done.

$$[A \mid I] \xrightarrow[E_k \cdots E_1]{\text{mult by}} [I \mid E_k \cdots E_1].$$

A few properties of $^{-1}$:

- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = \frac{1}{c} A^{-1}$
- $(AB)^{-1} = B^{-1} \cdot A^{-1}$
- $(A^t)^{-1} = (A^{-1})^t$

$\Rightarrow$ Any A this is invertible is a product of elementary matrices:

$$A^{-1} = E_k \cdots E_1 \quad \Rightarrow \quad A = (A^{-1})^{-1} = (E_k \cdots E_1)^{-1} = E_1^{-1} \cdots E_k^{-1}.$$

Ex: $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$: To row reduce:

		Result
swap:	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$
kill 1st column	$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$
Change sign	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
kill 2nd column	$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$\leftrightarrow$ - inverses

Matrix Transformations

Matrices also provide functions $\mathbb{R}^n \to \mathbb{R}^m$ given by

domain codomain

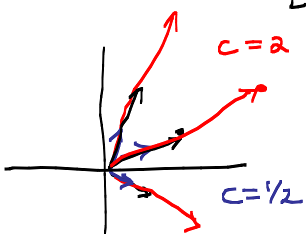
$$A(\vec{v}) = A\vec{v}$$

Many things we've studied have secretly been properties of these maps:

$A\vec{x} = \vec{b}$ has solutions $\longleftrightarrow$ $\vec{b}$ is in image of $A(\,)$.

Today we talked about 3 kinds of transformations from $\mathbb{R}^2$ to itself.

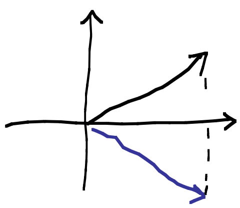
1) Dialations: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} cx \\ cy \end{bmatrix}$. If $|c|>1$, expansion; If $|c|<1$, contraction $\longleftrightarrow$ $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



So points expand away from origin: $|c|>1$

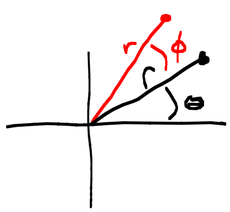
or they contract back towards it: $|c|<1$

2) Reflections: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix}$ etc. $\longleftrightarrow$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



reflect each point through the x-axis

3) Rotations:



$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ r\cos\theta\sin\phi + r\sin\theta\cos\phi \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\, x - \sin\phi\, y \\ \sin\phi\, x + \cos\phi\, y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$