At the end of last time, we defined linear combinations of a pair of vectors. Today we will use similar ideas to get a way to describe arbitrary vectors in terms of ones we like.

Example of what we want:

\[ \{e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1)\} \quad \text{in} \quad \mathbb{R}^3 \]

1) Every vector \((a,b,c)\) is a linear comb of these:

\[ (a,b,c) = ae_1 + be_2 + ce_3 \]

2) If \(ae_1 + be_2 + ce_3 = de_1 + fe_2 + ge_3\), then \(a=d, b=f, c=g\). Let each choice of coefs gives a different vector.

**Definition:** A set of vectors \(S = \{v_1, ..., v_n\}\) spans \(W\) if every vector in \(W\) is a linear comb of vectors of \(S\).

In other words, if \(w \in W\), then there are coefficients \(a_1, ..., a_n\) s.t. \(w = a_1v_1 + ...\)

**Example:**
- \(\{e_1, e_2, e_3\}\) spans \(\mathbb{R}^3\) (part i)
- \(\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\}\) spans the solutions to \(x+y+z=0\): \(\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}\)

\(\Rightarrow x\) is lead, \(y,z\) free \(\Rightarrow y=s, \quad z=t\)

\(x = -s-t \Rightarrow \) sol are of the form \((-s-t, s, t)\)
\[ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \]

- \{ (2,1,1), (0,2,1), (0,0,2) \} spans \( \mathbb{R}^3 \): have to show that we can find \( r,s,t \) s.t. \( r(2,1,1) + s(0,2,1) + t(0,0,2) = (a,b,c) \)

\[
\begin{bmatrix}
 2 & 0 & 0 \\
 0 & 1 & 0 \\
 1 & 2 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
 2a \\
 b \\
 c
\end{bmatrix}
= \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
 a/2 \\
 b/2 - a/4 \\
 c/2 - b/4 - a/8
\end{bmatrix}
\]

so \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a/2 \\ b/2 - a/4 \\ c/2 - b/4 - a/8 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \)

- \{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,1,1) \} spans \( \mathbb{R}^3 \)

Spanning means we have enough vectors to reach every point in our space. Doesn't say anything about uniqueness.

**Def** A set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is **linearly independent** if

\[ a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0} \Rightarrow a = b = c = 0. \]

(some def for bigger sets).

Put it in words, a set is lin ind if there is a unique way to write zero as a linear comb of the vectors. This actually implies that there is a unique way to write any vector in the span.

If \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is linearly independent, then

\[ a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = d\mathbf{\bar{v}}_1 + f\mathbf{\bar{v}}_2 + g\mathbf{\bar{v}}_3 \Rightarrow a = d, \ldots \]
why? Just subtract the right hand side from the left: 
\((a-d)\vec{v}_1 + (b-f)\vec{v}_2 + (c-g)\vec{v}_3 = 0\)

\[\text{Ex:}\ \{ (1,0), (0,1) \} \] is linearly independent:

\[a(1,0) + b(0,1) = (a,b) = (0,0) \iff a=b=0\]

\[-\{ (1,0,0), (0,1,0), (0,0,1), (1,1,1) \} \] is not linearly independent:

\[\bar{e}_1 + \bar{e}_2 + \bar{e}_3 - (1,1,1) = (0,0,0)\]

\[\bar{e}_1 + \bar{e}_2 + \bar{e}_3 = (1,1,1)\] so

A set is not linearly independent when one vector is a linear combination of the others.

Put together:

**Def** A set \(B\) is a **basis** if it is both linearly independent and spans.

**Ex:** \(\{ \bar{e}_1, \bar{e}_2, \bar{e}_3 \} \) is a basis for \(\mathbb{R}^3\). The standard basis.

\[-\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \} \] is a basis for the solutions to

\[x+y+z=0.\]

Already saw it spans. Why linearly independent?

\[a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]

\[\begin{bmatrix} a-b \\ b \\ a \end{bmatrix} \Rightarrow a=b=0\]
The size of any basis is the **dimension** of the space.

Ex: \( \dim \mathbb{R}^3 = 3 \)

- \( \dim (x+y+z=0^3) = 2 \)

**Remark:** This fits our intuitive, geometric notion of dimension: 1 dim things are lines; 2 dim things are planes, etc.

So if \( B \) is a basis, then any vector is a linear combination of vectors in \( B \) (and uniquely so). Finding the coefficients is tough.

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**Dot Product**

**Def** \( \overrightarrow{u} = (u_1, \ldots, u_n) \), \( \overrightarrow{v} = (v_1, \ldots, v_n) \), then

\[
\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + \ldots + u_nv_n
\]

Ex: *(1, 2) \cdot (3, 4) = 1(3) + 2(4) = 11*

* (1, 8, 16) \cdot (2, 0, 1) = 1\cdot2 + 8\cdot0 + 16\cdot1 = 18*

**Properties of \( \cdot \):**

1. \( \overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u} \)
2. \( \overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w} \)
3. \( (c\overrightarrow{u}) \cdot \overrightarrow{v} = c(\overrightarrow{u} \cdot \overrightarrow{v}) \)
4. \( \overrightarrow{u} \cdot \overrightarrow{u} \geq 0 \), w/ equality iff \( \overrightarrow{u} = \overrightarrow{0} \).

**How to see these? Expand out:**

1. \( \overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = u_1(v_1 + w_1) + \ldots + u_n(v_n + w_n) \)

\[
= u_1v_1 + u_1w_1 + \ldots + u_nv_n + u_dw_n
\]

\[
= (u_1v_1 + \ldots + u_nv_n) + (u_1w_1 + \ldots + u_dw_n) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}
\]

**Part 4:** Says that we can use \( \cdot \) to get lengths:

**Def** The **length / norm** of \( \overrightarrow{u} \), \( |\overrightarrow{u}| \) is

\[
|\overrightarrow{u}| = \sqrt{\overrightarrow{u} \cdot \overrightarrow{u}} = \sqrt{u_1^2 + \ldots + u_n^2}
\]
Ex: \( \vec{u} = (3, 4) \), \( |\vec{u}| = \sqrt{9 + 16} = 5 \)
- \( \vec{v} = (3, 4, 12) \), \( |\vec{v}| = \sqrt{9 + 16 + 144} = \sqrt{169} = 13 \)

**Def:** A **unit vector** is any vector of length 1.

Given any vector, we can normalize to get a unit vector: \( \vec{u}_u = \frac{\vec{u}}{|\vec{u}|} \)

Ex: \( \vec{u} = (3, -4, 12) \), \( |\vec{u}| = 13 \) \( \implies \vec{u}_u = \left( \frac{3}{13}, \frac{-4}{13}, \frac{12}{13} \right) \)

The unit vectors measure just direction, rather than length.

Can get more geometry from \( \cdot \):

**Def:** The **angle** between two vectors \( \vec{u} \) and \( \vec{v} \) is defined by

\[
\cos \Theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}, \quad 0 \leq \Theta \leq \pi
\]

**Def:** Two vectors are **orthogonal** if \( \Theta = \pi/2 \) \( \iff \vec{u} \cdot \vec{v} = 0 \)

Ex: i) \( \vec{u} = (3, 1) \), \( \vec{v} = (-1, 3) \) are orthogonal
ii) \( \vec{u} = (0, 1) \), \( \vec{v} = \left( \frac{1}{2}, \sqrt{3}/2 \right) \)

\[
\cos \Theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{1}{1 \cdot 1} = \frac{\sqrt{3}}{2} \implies \Theta = \pi/6
\]

With these notions, the linear algebra captures many of the ideas we had from ordinary geometry:

**Distance between points:**

\[
d(\vec{u}, \vec{v}) = |\vec{u} - \vec{v}| = \sqrt{(u_1 - v_1)^2 + \ldots}
\]

This gives us things like circles:

\[
\{ \vec{x} \mid d(\vec{x}; \vec{a}) = r \}
\]