**Def** A linear transformation is **1-1** (or injective) if

\[ L(v) = L(w) \Rightarrow v = w. \]

The vector space structure makes it easier for us to check this:

**Prop** \( L: V \rightarrow W \) is 1-1 \( \iff \ker(L) = \{0\}. \)

**Prf** \( L \) 1-1 \( \Rightarrow \) If \( \overline{v} \in \ker(L) \), then \( L(\overline{v}) = \overline{0} = L(\overline{0}) \Rightarrow \overline{v} = \overline{0} \) (by 1-1).

If \( L(\overline{v}) = L(\overline{w}) \), then \( \overline{0} = L(\overline{v}) - L(\overline{w}) = L(\overline{v} - \overline{w}) \Rightarrow \overline{v} - \overline{w} \in \ker(L) \)

So if \( \ker(L) = \{0\} \), then \( L(\overline{v}) = L(\overline{w}) \Rightarrow \overline{v} - \overline{w} = \overline{0} \Rightarrow \overline{v} = \overline{w}. \)

The argument actually shows that any two \( \overline{v}, \overline{w} \) s.t. \( L(\overline{v}) = L(\overline{w}) \) differ by an element of \( \ker(L) \). We'll return to this.

**Ex:** \( L_2: P_2(x) \rightarrow P_3(x) \), \( L_2(p) = \int_0^x p(t) \, dt \) from last time. Saw \( \ker(L_2) = \{0\} \), so \( L_2 \) is injective. Also see by direct computation that \( \mathcal{E} \langle x, x^2, x^2 \rangle \rightarrow \mathcal{E} \langle x_1, x_2, x_1^2 \rangle \) which is lin. nd., though not a basis.

**1-1 transforms preserve lin. ind.**

**Prf** If \( L \) is 1-1 & \( \mathcal{E} \overline{x}_1, \ldots, \overline{x}_k \) is lin. nd., then \( \mathcal{E} L(\overline{x}_1), \ldots, L(\overline{x}_k) \) is lin. ind.

**Prf** Look at \( a_1 L(\overline{x}_1) + \cdots + a_k L(\overline{x}_k) = \overline{0} \)

\[ L(a_1 \overline{x}_1 + \cdots + a_k \overline{x}_k) = \overline{0} \]

1-1 \( \Rightarrow \) \( a_1 \overline{x}_1 + \cdots + a_k \overline{x}_k = \overline{0} \)

\( \mathcal{E} \overline{x}_1, \ldots, \overline{x}_k \) lin. ind \( \Rightarrow \) \( a_1 = \cdots = a_k = 0. \)

So a 1-1 transform preserves information in \( V \).

**Companion topic is onto.**

**Def** \( L: V \rightarrow W \) is **onto** (surjective) if \( \text{Range}(L) = W \).
Ex: $L_1: P_2(x) \rightarrow P_1(x)$, $L_1(p) = p(x)$. Then $L_1$ is surjective. Also see that $x^3, x, x^2 \in \mathbb{R}$ which isn't lin ind (0 is sol), but does span.

Thus we have a companion to the previous prop:

Prop If $L: V \rightarrow W$ is onto & $\exists x_1, \ldots, x_n \in V$ spans $V$, then $L(x_1), \ldots, L(x_n)$ spans $W$.

An onto transformation sees all of the information in $W$.

Together they say that $V$ and $W$ are essentially the same.

Def A linear transformation $S: V \rightarrow W$ is **invertible** if there is a $T: W \rightarrow V$ s.t.

$$S(T(w)) = w \quad \text{for all } w \in W \text{ and }$$

$$T(S(v)) = v \quad \text{for all } v \in V.$$

This is also called an **isomorphism** and $V$ & $W$ are said to be **isomorphic**.

Ex: $T: \mathbb{R}^2 \rightarrow P_1(x)$ is invertible. Let $S: P_1(x) \rightarrow \mathbb{R}^2$

$$S(ax+b) = (a, b).$$

Thus $S(T(a, b)) = S(ax+b) = (a, b)$

$$T(S(ax+b)) = T(a, b) = ax+b.$$

Prop $T$ is invertible iff $T$ is 1-1 and onto.

So an isomorphism is an identification of $V$ with $W$ in a way that preserves all structure.

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General Systems of equations.

If $L: V \rightarrow W$ is a linear transformation, then any equation of the form

$$L(x) = \overline{b}$$

Is a system of equations.

This is consistent with the $\mathbb{R}^n$ case. Here

$$L(x) = \overline{b} \iff A \bar{x} = \overline{b} \quad \text{some } A \text{ specified by } L.$$

Our intuition from this case tells us what we should expect in general.

Prop: Any two solutions to $L(x) = \overline{b}$ differ by an element of the null space $\text{null}(L)$.

**PF** if $L(\bar{x}) = \overline{b}$ and $L(\bar{w}) = \overline{b}$, then $L(\bar{x} - \bar{w}) = L(\bar{x}) - L(\bar{w}) = \overline{b} - \overline{b} = \overline{0}$.

This helps us make sense of null space from other classes like:

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Ex. Let $D: \mathcal{C}^0(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$ be differentiation. Then any solution to a system like
\[
(D^2 - 2D - 3\text{Id})(f) = 2x
\]
\[
f''(x) - 2f'(x) - 3f(x)
\]
is of the form
\[
y = y_h + y_p,
\]
where
\[
y_p
\]
is some fixed sol and $y_h$ is a sol to $f'' - 2f' - 3f = 0$.

(In this case $\ker(D^2 - 2D - 3\text{Id}) = \text{Span}(e^{-x}, e^{3x})$, and a choice of $y_p$ is given by
\[
-(\frac{2}{3})x + \frac{1}{9}
\].)

Our previous prop. shows this holds in general.

Prop. If $\mathcal{V}$ is a solution to $L(x) = 0$, then $\mathcal{V} = \mathcal{V}_h + \mathcal{V}_p$ for some fixed sol $\mathcal{V}_p$ and some $\mathcal{V}_h \in \ker(L)$.

Thus all solutions are of the form $\mathcal{V}_p + \mathcal{V}_h$, so $\{\text{Sols}\} = \{\mathcal{V}_p + \mathcal{V}_h | \mathcal{V}_h \in \ker(L)\} = \mathcal{V}_p + \ker(L)$. 

by def