Lecture 20 – Linear Transformations, Kernel & Image

Note Title 4/2/2008

Def If V and W are vector spaces, then a function

$$L: V \to W \quad \text{is a } \textit{linear transformation}$$

if 1) $L(\bar{u}+\bar{v}) = L(\bar{u}) + L(\bar{v})$

2) $L(\alpha\bar{v}) = \alpha L(\bar{v})$.

This is very like the notion of a subspace:

subspace { closed under addition ⟷ sums go to sums }
{ closed under scalar mult ⟷ products to products } linear transformation

non-empty (contains $\bar{0}$) ⟵ Morally Same ⟶ "is defined"

Ex: $V = P_2(x)$, $W = P_3(x)$

$L_1 = \frac{d}{dx}$

$L_1(ax^2+bx+c) = 2ax+b$. The derivative is linear. This we know from calculus.

Direct check: $p(x) = ax^2+bx+c$, $q(x) = rx^2+sx+t$

$$(p+q)(x) = (a+r)x^2 + (b+s)x + (c+t)$$

$$L_1((p+q)(x)) = 2(a+r)x + (b+s) = (2ax+b) + (2rx+s) = L_1(p) + L_1(q)$$

$$L_1(\alpha p(x)) = 2\alpha a x + \alpha b = \alpha(2ax+b) = \alpha L_1(p)$$

$L_2 = \int_0^x$

$L_2(p(x)) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$. Integration is linear. This we know from calculus.

Direct check:

$$L_2((p+q)(x)) = \frac{(a+r)}{3}x^3 + \frac{(b+s)}{2}x^2 + (c+t)x = \left(\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right) + \left(\frac{r}{3}x^3+\frac{s}{2}x^2+tx\right) = L_2(p)+L_2(q)$$

$$L_2(\alpha p(x)) = \frac{\alpha a}{3}x^3 + \frac{\alpha b}{2}x^2 + \alpha c x = \alpha\left(\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right) = \alpha L_2(p(x)).$$

Remark: L is linear if and only if $L(a\bar{v}+b\bar{w}) = aL(\bar{v}) + bL(\bar{w})$.

Def If $L: V\to W$ is linear, the *kernel* of L is

$$\ker(L) = \{\bar{v}\in V \mid L(\bar{v}) = \bar{0}\}.$$

Prop $\ker(L)$ is a subspace.

1) $\bar{0}\in\ker(L)$: $\quad \bar{0} = 0\cdot\bar{0} \Rightarrow L(\bar{0}) = L(0\cdot\bar{0}) = 0\cdot L(\bar{0}) = \bar{0}$.

$$\begin{array}{ccc} 2)\ \bar v,\bar w \in \ker(L) & \longleftrightarrow & L(\bar v)=L(\bar w)=\bar 0 \\ \Downarrow ? & & \Downarrow ? \\ \bar v+\bar w \in \ker(L) & \longleftrightarrow & L(\bar v+\bar w)=\bar 0 \end{array} \qquad L(\bar v+\bar w)=L(\bar v)+L(\bar w)=\bar 0+\bar 0=\bar 0. \quad \checkmark$$

$$\begin{array}{ccc} 3)\ \bar v \in \ker(L) & \longleftrightarrow & L(\bar v)=\bar 0 \\ \Downarrow ? & & \Downarrow ? \\ a\bar v \in \ker(L) & \longleftrightarrow & L(a\bar v)=\bar 0 \end{array} \qquad L(a\bar v)=aL(\bar v)=a\cdot\bar 0=\bar 0. \quad \checkmark$$

Def The nullity is the dimension of $\ker(L)$.

If $V=\mathbb{R}^n$, $W=\mathbb{R}^m$

$$\begin{array}{ccc} L: V \longrightarrow W \text{ linear} & & L_A(\bar v)=A\bar v \\ \updownarrow & & \uparrow \\ A \in M_{m\times n} & & A \end{array}$$

$\ker(L_A)=\{\bar v \mid L_A(\bar v)=\bar 0\}=\{\bar v \mid A\bar v=\bar 0\}=$ solutions to the homogeneous system $A\bar x=\bar 0$.

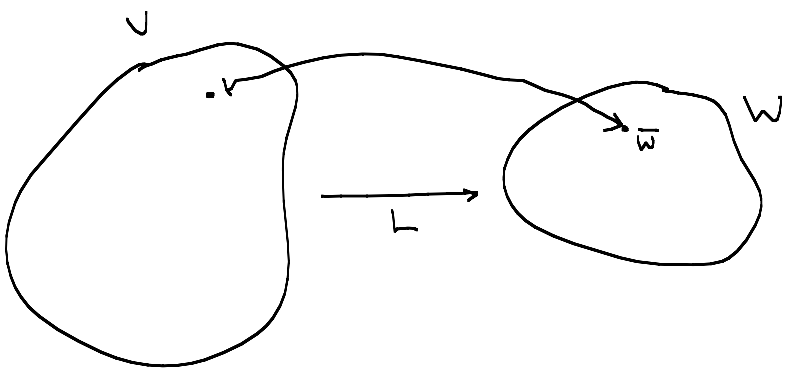
So the nullity is the dimension of the space of solutions = # of free variables.

Ex L_1 as above. $\ker(L_1)=\{p \mid p'(x)=0\}=\{ax^2+bx+c \mid 2ax+b=0\}=\{c\}=P_0(x)$.
$\Rightarrow a=b=0$

L_2 as above. $\ker(L_2)=\{p \mid \int_0^x p=0\}=\{ax^2+bx+c \mid \frac{a}{3}x^3+\frac{b}{2}x^2+cx=0\}=\{0\}$.
$\Rightarrow a=b=c=0$

Def The range or image of $L: V\longrightarrow W$ is the set of all $\bar w\in W$ s.t. there exists a $\bar v\in V$ with $L(\bar v)=\bar w$.

In pictures:



Prop $\mathrm{Im}(L)$ is a subspace.

Pf: 1) $\bar 0\in \mathrm{Im}(L)$. (Saw before)

$$\begin{array}{ccc} 2)\ \bar w_1,\bar w_2 \in \mathrm{Im}(L) & \longleftrightarrow & \text{Can find } \bar v_1,\bar v_2 \text{ s.t. } L(\bar v_1)=\bar w_1,\ L(\bar v_2)=\bar w_2 \\ \Downarrow ? & & \Downarrow ? \\ a\bar w_1+b\bar w_2 \in \mathrm{Im}(L) & \longleftrightarrow & \text{Can find } \bar v \text{ s.t. } L(\bar v)=a\bar w_1+b\bar w_2. \end{array}$$

$a\bar w_1+b\bar w_2 = aL(\bar v_1)+bL(\bar v_2)=L(a\bar v_1+b\bar v_2)$, so $\bar v=a\bar v_1+b\bar v_2$ works. $\boxtimes$

If $V=\mathbb{R}^n$, $W=\mathbb{R}^m$, then

$$\mathrm{Im}(L_A) = \{\bar{w} \in \mathbb{R}^m \mid L_A(\bar{v}) = \bar{w}, \text{ some } \bar{v}\} = \{\bar{w} \mid A\bar{v} = \bar{w}, \text{ some } \bar{v}\}$$
$$= \{\bar{w} \mid A\bar{v} = \bar{w} \text{ has solutions}\}$$
$$= \{\bar{w} \mid \bar{w} \in \text{columnspace}(A)\}$$
$$= \text{Column Space of } A.$$

So $\dim \mathrm{Im}(L_A) = \mathrm{rank}(A) = \#$ lead variables.

__Def__ The __rank__ of L is $\dim \mathrm{Im}(L_A)$.

Thm (Rank-Nullity): $L: V \longrightarrow W$ linear, then

$$\mathrm{rank}(L) + \mathrm{nullity}(L) = \dim V.$$

for $\mathbb{R}^n$, etc case, this just says "variables are either free or lead".