Lecture 18 - Basis & Rank

Starting question: how can we recognize if we have a basis?

1) Another formulation: \( \sum \overline{v}_i \) is a basis if for any \( \overline{v} \in V \), there are unique numbers \( a_1, \ldots, a_n \) s.t.

\[ \overline{v} = a_1 \overline{v}_1 + \ldots + a_n \overline{v}_n \]

If we think of this as a system with variables \( a_1, \ldots, a_n \), then we are asking when the system has a unique solution.

\( \Rightarrow \) often see span & linear independence at the same time: write out system and look at RE form.

2) If \( \dim V = n \), then any lin ind set w/ \( n \) vectors is a basis.

3) any spanning set w/ \( n \) vectors is a basis.

Put together: \( a_1 \overline{v}_1 + \ldots + a_n \overline{v}_n = \overline{v} \) normally becomes a system: \( A\overline{x} = \overline{b} \) if \( |A| \neq 0 \), then \( \sum \overline{v}_i, \ldots, \overline{v}_n \) is a basis.

Ex: \( \{1, x+x^2, x-x^2\} \) is a basis for \( P_2 \)

System: \( a + b(x+x^2) + c(x-x^2) = a_0 + a_1 x + a_2 x^2 \)

or

\[
\begin{align*}
a & = a_0 \\
b + c & = a_1 \\
b - c & = a_2
\end{align*}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}
\]

\[ |A| = 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -2 \neq 0 \Rightarrow \text{basis.} \]

Rank: This we could have done long ago.

Recall: If \( A \) is an \( m \times n \) matrix then \( A = [\overline{a}_1, \ldots, \overline{a}_n] \) , \( \overline{a}_i \in \mathbb{R}^m \), &

\[
A = [\overline{a}_1 \ldots \overline{a}_m], \quad \overline{a}_i \in \mathbb{R}^n.
\]

Def: The row rank of \( A \) is \( \text{dim } \text{span}(\overline{a}_1, \ldots, \overline{a}_m) \).

The column rank of \( A \) is \( \text{dim } \text{span}(\overline{a}_1, \ldots, \overline{a}_n) \).
Thm. These are always equal: rank
So we can take our favorite: row rank comes from RREF form

- column rank tells us about sol to \( AX = \mathbf{b} \).

If \( A \) and \( B \) are row equivalent, then \( \text{rank}(A) = \text{rank}(B) \): rows of \( B \) are linear combos of rows of \( A \) and vice versa.

So \( \text{rank}(A) = \text{rank}(\text{RREF form of } A) = \text{rank} (\text{RREF of } A) \)

This last quantity is easy to see: it is the number of non-zero rows.

Ex:
\[
\begin{bmatrix}
1 & 2 & 4 & 6 \\
2 & 4 & 9 & 13 \\
3 & 6 & 13 & 19
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 4 & 6 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\([1 \ 2 \ 0 \ 2]\) and \([0 \ 0 \ 1 \ 1]\) are lin ind \((1^{\text{st}} \text{ and } 3^{\text{rd}} \text{ coord show this}) \Rightarrow \text{rank} = 2.\)

Now the geometric story:
you showed that \( AX = \mathbf{b} \) has solutions iff \( \mathbf{b} \) is a linear comb of the columns of \( A \leftrightarrow \mathbf{b} \) in the column space
So \( \text{rank}(A) \) tells us how many \( \mathbf{b} \) work. Now can ask for a basis:

RREF form again.

2 ways:
I. Find the \( \text{RREF of } A^t \). Row space \( \text{(A^t)} = \text{column space (A)} \).
II. Find the \( \text{RREF of A}. \) The basis of \( \text{column space (A)} \) is given by those columns with a leading 1 in the \( \text{RREF form} \).

Ex:
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
4 & 9 & 13 \\
6 & 13 & 19
\end{bmatrix}
\]

\( A^t = \) matrix above, and saw \( \begin{bmatrix}1
\end{bmatrix}, \begin{bmatrix}0
\end{bmatrix} \) forms a basis.

Now can go the other direction: If \( \mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n \) can use this idea to find a basis for the span:
\[ A = \begin{bmatrix} \frac{v_1}{v_b} \\ \vdots \\ \frac{v_r}{v_b} \end{bmatrix} \quad \rightarrow \quad \text{REF}(A) \quad \rightarrow \quad \text{basis.} \]