Lecture 18 – Basis & Rank

Note Title 3/24/2008

Starting question: how can we recognize if we have a basis?

1) Another formulation: $\{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis if for any $\bar{v} \in V$, there are <u>unique</u> numbers $a_1, \dots, a_n$ s.t.

$$\bar{v} = a_1\bar{v}_1 + \dots + a_n\bar{v}_n$$

<u>If</u> we think of this as a system with variables $a_1, \dots, a_n$, then we are asking when the system has a unique solution.

$\Rightarrow$ often see span & linear independence at the same time: write out system and look at RE form.

2) If $\dim V = n$, then • any lin ind set w/ n vectors is a basis.

• any spanning set w/ n vectors is a basis.

Put together: $a_1\bar{v}_1 + \dots + a_n\bar{v}_n = \bar{v}$ normally becomes a system: $A\bar{x} = \bar{b}$ if $|A| \neq 0$, then $\{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis.

<u>Ex</u>: $\{1, x+x^2, x-x^2\}$ is a basis for P_2

System: $a + b(x+x^2) + c(x-x^2) = a_0 + a_1x + a_2x^2$

or

$$\begin{aligned} a \quad\quad\quad &= a_0 \\ b + c &= a_1 \\ b - c &= a_2 \end{aligned} \quad \longleftrightarrow \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$|A| = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0. \Rightarrow \text{basis.}$$

<u>Rank</u> This we could have done long ago.

<u>Recall</u>: If A is an $m \times n$ matrix then $A = [\bar{a}_1 | \cdots | \bar{a}_n]$, $\bar{a}_i \in \mathbb{R}^m$, & $A = \begin{bmatrix} \bar{a}^1 \\ \vdots \\ \bar{a}^m \end{bmatrix}$, $\bar{a}^i \in \mathbb{R}^n$.

<u>Def</u> The <u>row rank</u> of A is $\dim \overbrace{\mathrm{span}(\bar{a}^1, \dots, \bar{a}^m)}^{\text{row space}}$.

The <u>column rank</u> of A is $\dim \underbrace{\mathrm{span}(\bar{a}_1, \dots, \bar{a}_n)}_{\text{column space}}$.

Thm These are always equal: rank

So we can take our favorite: • row rank comes from RE form

• column rank tells us about sol to $A\bar{x}=\bar{b}$.

If A and B are row equivalent, then $\text{rank}(A)=\text{rank}(B)$: rows of B are linear combs of rows of A and vice versa.

So $\text{rank}(A) = \text{rank}(\text{RE form of } A) = \text{rank}(\text{RREF of } A)$

This last quantity is easy to see: It is the number of non-zero rows.

Ex: $A = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 4 & 9 & 13 \\ 3 & 6 & 13 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$[1\ 2\ 0\ 2]$ and $[0\ 0\ 1\ 1]$ are lin ind (1^{st} and 3^{rd} coord show this) $\Rightarrow$ rank $= 2$.

Now the geometric story:

You showed that $A\bar{x}=\bar{b}$ has solutions iff $\bar{b}$ is a linear comb of the columns of A $\longleftrightarrow$ $\bar{b}$ in the column space

So $\text{rank}(A)$ tells us how many $\bar{b}$ work. Now can ask for a basis: RRE form again.

2 ways:

I. Find the RREF of A^t. Row Space(A^t) = column space(A).

II. Find the RREF of A. The basis of column space(A) is given by those columns with a leading 1 in the RRE form.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 9 & 13 \\ 6 & 13 & 19 \end{bmatrix}$ A^t = matrix above, and saw $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ forms a basis.

Now can go the other direction: If $\bar{v}_1, \dots, \bar{v}_k \in \mathbb{R}^n$, can use this idea to find a basis for the span:

$$A = \begin{bmatrix} \overline{v}_1 \\ \vdots \\ \overline{v}_k \end{bmatrix} \rightsquigarrow \text{RRE form}(A) \rightsquigarrow \text{basis.}$$