Lecture 13 - Determinant & Inverses

Last time finished with properties of the determinant. In particular, we saw easy rules to understand what happens when we apply elementary operations. Two important consequences:

1) If we repeat a row or column, the det = 0.
2) If one row is a scalar multiple of another, then the det = 0.

Ex: \[
\begin{bmatrix}
73 & 81 & 9 & 6 \\
14 & 97 & -1 & 0 \\
6 & 8 & 5 & 3 \\
12 & 16 & 10 & 6
\end{bmatrix}
\]

Row₄ = 2 Row₃ \Rightarrow \det (A) = 0.

Why? 1) \Rightarrow 2) by factoring out the scalar.

1) Follows because swapping the equal rows leaves the matrix fixed while swapping the sign of the det.

Can further simplify our lives by performing a few row operations:

If \( U \) is upper triangular, then \( \det(U) = \) product of diagonal entries.

So we can find \( \det(A) \) by row reducing down to an upper triangular matrix:

- adding rows to other rows does nothing
- swapping rows swaps sign:

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & -2 \\
0 & -2 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & -2 \\
0 & 0 & -2
\end{bmatrix}
\]

\[
|A| = \frac{1}{2} \quad |\begin{vmatrix}
1 & 1 & 1 \\
0 & 0 & -2 \\
0 & 0 & -2
\end{vmatrix}| = -1 \cdot (-2) \cdot (-2) = 4
\]

This is a relatively useful method for finding \( |A| \).

Det has 1 more property: “linear in each row”:

\[
A = \begin{bmatrix}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_n
\end{bmatrix}.
\]
If \( B = \begin{bmatrix} b_1 & \ldots & b_n \\ \vdots & \ddots & \vdots \\ b_m & \ldots & b_n \end{bmatrix} \), then
\[
|B| = \begin{vmatrix} b_1 & \ldots & b_n \\ \vdots & \ddots & \vdots \\ b_1' & \ldots & b_n' \end{vmatrix} + \begin{vmatrix} b_2' & \ldots & b_m' \\ \vdots & \ddots & \vdots \\ b_1 & \ldots & b_n \end{vmatrix}.
\]
This, together with "repeated rows give zero" and \(|I| = 1\) determine \( \det \).

Last time finished with: \( \text{A invertible } \Rightarrow \det(A) \neq 0 \).

Today we will show the converse and give a formula for \( A^{-1} \) if \( \det(A) \neq 0 \).

Def If \( A \) is \( n \times n \), the \textbf{matrix of cofactors} is the \( n \times n \) matrix of cofactors of elements of \( A \):
\[
C = \begin{bmatrix} C_{11} & \ldots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \ldots & C_{nn} \end{bmatrix}
\]

The \textbf{adjoint} of \( A \), \( \text{adj}(A) \) is defined by \( \text{adj}(A) = C^t \).

The importance of \( \text{adj}(A) \) is the following:
\[
A \cdot \text{adj}(A) = |A|I
\]

\textbf{Proof}:
The \((i,j)\)th entry of \( A \cdot \text{adj}(A) \) is row \( i \) of \( A \) \cdot column \( j \) of \( \text{adj}(A) \):
\[
[a_{i1} \ldots a_{in}] \cdot \begin{bmatrix} C_{1j} \\ \vdots \\ C_{nj} \end{bmatrix} = a_{i1} C_{1j} + \ldots + a_{in} C_{nj}.
\]
For \( i = j \), this is \( a_{i1} C_{1j} + \ldots + a_{in} C_{nj} = |A| \).
For \( i \neq j \), this is the determinant of the matrix we get by replacing row \( j \) with a copy of row \( i \). \( \Rightarrow \) for \( i \neq j \), this sum is 0!

What do we get?
\[
\text{If } |A| \neq 0, \text{ then } A^{-1} = \frac{1}{|A|} \text{adj}(A).
\]
This is a beautiful, theoretical result that holds very generally! In particular, if the entries of \( A \) are integers and \( |A| = \pm 1 \), then the entries of \( A^{-1} \) are integers!

\textbf{Ex}:
\[
A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad |A| = 3 - 2 = 1, \quad C_{11} = 1, \quad C_{12} = -1, \quad C_{21} = -2, \quad C_{22} = 3 \quad \Rightarrow
\]
\[
C = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \quad \Rightarrow \quad \text{adj}(A) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}
\]

Also gives another test for linear independence:
\( \text{if } \vec{v}_1, \ldots, \vec{v}_n \text{ in } \mathbb{R}^3 \text{ is lin ind } \text{ if and only if } |\vec{v}_1 \ldots \vec{v}_n| \neq 0 \).
Better stated:

$$AX = B$$ has a unique solution iff $$|A| \neq 0$$.

This is actually a formula for finding the inverse in terms of the entries of A:

Cramer's Rule.

Let $$B_i$$ be the matrix we get by replacing column $$i$$ of A with B. Then

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \text{ where } x_i = \frac{|B_i|}{|A|}.$$