Introduction

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Syllabus / course policy are online. A few things of note:
1) Computational Homeworks are online via Webwork
2) In the next few weeks, we'll roll out online resources
   - class blog - you can log on & make posts
   - class wiki - your versions of defns, etc from class
3) Homeworks, exams, etc are the same across all sections.

Linear Systems / Matrices

Def: A linear system (= a system of equations) in 3 variables is a collection of equations
\[ a_1x + b_1y + c_1z = d_1 \]
\[ \vdots \]
\[ a_nx + b_ny + c_nz = d_n \]

A solution is a point \((x, y, z)\) making every equation true.

Geometric story: \(ax + by + cz = d\) is the equation of a plane in \(\mathbb{R}^3\). So points making this true \((x, y, z)\) = points on the plane.

\(\Rightarrow\) solutions to a system are those points sitting on every plane in the system. = points where they intersect.
3 cases:

1) They don't all simultaneously intersect

3 planes, 2 are parallel
⇒ don't intersect

Say the system is inconsistent

Otherwise the system is consistent

2) They intersect at exactly one point

Say the system is independent

3) They intersect in a line or a plane

(from the top)

Say the system is underdetermined/dependent

Note that \(x, y, z,\) etc. are basically placeholders.
\(x = \) first spot, \(y = 2^{nd}\) spot, etc.
To make things easier on ourselves (and comps),
we make some new notation.
Def A matrix is a rectangular array of numbers.

\[
\begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{m1} & \cdots & a_{mn}
\end{bmatrix}
\]

\[\text{rows} \quad \text{columns}\]

To a system of equations, we can associate a matrix of coefficients by just dropping the variables:
(If a variable doesn’t occur, its coef is 0)

\[\begin{align*}
3x + 17y + 21z &= 146 \\
2x + 13z &= 19 \\
y + z &= 5
\end{align*}\]

\[
\begin{bmatrix}
    3 & 17 & 21 \\
    2 & 0 & 13 \\
    0 & 1 & 1
\end{bmatrix}
\]

Can also associate an augmented matrix

\[
\begin{bmatrix}
    3 & 17 & 21 & | & 146 \\
    2 & 0 & 13 & | & 19 \\
    0 & 1 & 1 & | & 5
\end{bmatrix}
\]

(Added the column of “=” to the coeff matrix)

More: Anything we can say about a system is reflected in the augmented matrix.

Q: How do we find solutions?

Lots of ways:

1) Graph everything
2) Substitute in
3) Make the system easier
Example: \( x + 2y = 4 \)  
\( 2x + y = 5 \) 

Substitute:

\[
(1) \iff x = 4 - 2y \\
(2) \Rightarrow 2(4 - 2y) + y = 5 \\
\Rightarrow 8 - 3y = 5 \\
\Rightarrow y = 1 \Rightarrow x = 2
\]

What makes a system easy?

- Each variable occurs once
- Its coefficient is 1.

\[
x = \frac{3}{2} \quad y = 1 \
\]

What can we do? Elementary operations

1. Can reorder the eqns / row
2. Can scale an eqn / row by a non-zero number.
3. Can add a multiple of one eqn / row to another (why does this not change things?)

Example: System above: \( x + 2y = 4 \)  
\( 2x + y = 5 \)  
\[
\begin{bmatrix}
1 & 2 & 4 \\
2 & 1 & 5
\end{bmatrix}
\]

Add \((-2) \cdot \text{eqn}_1\) to \text{eqn}_2:

\[
\begin{cases}
x + 2y = 4 \\
-3y = -3
\end{cases}
\]

Divide \text{eqn}_2\) by \(-3\):

\[
\begin{cases}
x + 2y = 4 \\
y = 1
\end{cases}
\]

Add \((-2) \cdot \text{eqn}_2\) to \text{eqn}_1:

\[
\begin{cases}
x = 2 \\
y = 1
\end{cases}
\]

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & -3 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & -3 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix}
\]
Def A matrix is in row echelon form if
1) each row starts with more zeros than the previous
2) the 1st non-zero entry in each row is a 1.

Ex: \[
\begin{bmatrix}
1 & 2 & 3 & 19 \\
0 & 1 & 4 & 6
\end{bmatrix}
\] is in RE form

- \[
\begin{bmatrix}
0 & 1 & 3
\end{bmatrix}
\] is not (fails 1)

- \[
\begin{bmatrix}
0 & 2 & 3
\end{bmatrix}
\] is not (fails 2)

Def A matrix is in reduced row echelon form if in addition
3) the 1st non-zero entry in each row is the only non-zero entry in its column.

Ex: \[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] Don't matter!

Augmented matrices in RREF are very easy to solve:
- variables corresponding to the start of rows occur exactly once
- can solve for these in terms of the remaining variables

In row echelon form, it is easy to see how many solutions a system has:

Ex: \[
\begin{bmatrix}
1 & \star & \star & \star \mid \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & 1
\end{bmatrix}
\] tells us \( x \), uniquely
\[
\begin{bmatrix}
1 & \star & \star & \star \mid \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & 1
\end{bmatrix}
\] tells us \( y \), uniquely
\[
\begin{bmatrix}
1 & \star & \star & \star \mid \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & 1
\end{bmatrix}
\] tells us \( z \), uniquely
General case: If the RE form has 1s along the diagonal (those entries where the row is the same as the column), then there is a unique solution to the system.

Remark: The condition that each row has more leading zeros than the previous forces the 1st possible non-zero position to be on the diagonal.

Ex: \[
\begin{bmatrix}
1 & * & * & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & a
\end{bmatrix} - \text{this translates to}
\]
\[
0 \cdot x + 0 \cdot y + 0 \cdot z = a
\]

So we see that if \( a \neq 0 \), then the system is inconsistent. The RE form shows us this in general with a row like \[
\begin{bmatrix}
0 & \cdots & 0 & 1 \cdot a
\end{bmatrix}, \quad a \neq 0.
\]