

APMA 308

HW #9

5.4: 2; 3a, b; 8

4.1: 1; 8; 11a; 18b, d; 20a, b; 27b, c; 30

4.2: 2b, e; 5b; 15b;

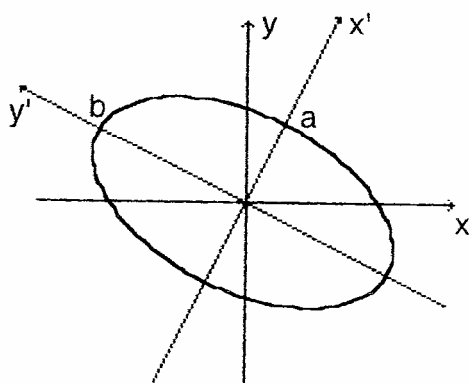
5.4

2. (a) $11x^2 + 4xy + 14y^2 - 60 = [x \ y] \begin{bmatrix} 11 & 2 \\ 2 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 60$. From Exercise 6(b) in

Section 5.3, $C^t \begin{bmatrix} 11 & 2 \\ 2 & 14 \end{bmatrix} C = \begin{bmatrix} 15 & 0 \\ 0 & 10 \end{bmatrix}$, where $C = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$, so the given

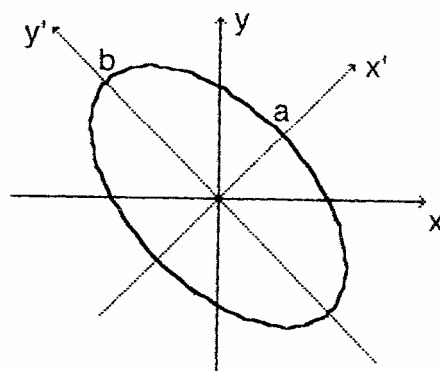
equation becomes $[x \ y] C \begin{bmatrix} 15 & 0 \\ 0 & 10 \end{bmatrix} C^t \begin{bmatrix} x \\ y \end{bmatrix} - 60 = 0$, or $[x' \ y'] \begin{bmatrix} 15 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - 60 = 0$,

where $\begin{bmatrix} x' \\ y' \end{bmatrix} = C^t \begin{bmatrix} x \\ y \end{bmatrix}$. Thus $15x'^2 + 10y'^2 = 60$; i.e., $\frac{x'^2}{4} + \frac{y'^2}{6} = 1$. The graph is an ellipse with the lines $y = 2x$ and $x = -2y$ as axes.



$$a = 2, b = \sqrt{6}$$

Figure for 2(a)



$$a = \sqrt{3}, b = \sqrt{6}$$

Figure for 2(b)

- (b) $3x^2 + 2xy + 3y^2 - 12 = [x \ y] \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 12$. From Exercise 6(c) in Section

5.3, $C^t \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, where $C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, so the given equation

becomes $[x \ y] C \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} C^t \begin{bmatrix} x \\ y \end{bmatrix} - 12 = 0$, or $[x' \ y'] \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - 12 = 0$,

where $\begin{bmatrix} x' \\ y' \end{bmatrix} = C^t \begin{bmatrix} x \\ y \end{bmatrix}$. Thus $4x'^2 + 2y'^2 = 12$; i.e., $\frac{x'^2}{3} + \frac{y'^2}{6} = 1$. The graph is an ellipse with the lines $y = x$ and $y = -x$ as axes.

- (c) $x^2 - 6xy + y^2 - 8 = [x \ y] \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 8$. The symmetric matrix has eigenvalues $\lambda = -2$ and $\lambda = 4$ with corresponding orthonormal eigenvectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ so } C^t \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} C = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}, \text{ where } C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ so}$$

the given equation becomes $[x \ y] C \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} C^t \begin{bmatrix} x \\ y \end{bmatrix} - 8 = 0$, or

$$[x' \ y'] \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - 8 = 0, \text{ where } \begin{bmatrix} x' \\ y' \end{bmatrix} = C^t \begin{bmatrix} x \\ y \end{bmatrix}. \text{ Thus } -2x'^2 + 4y'^2 = 8; \text{ i.e.,}$$

$$\frac{-x'^2}{4} + \frac{y'^2}{2} = 1. \text{ The graph is a hyperbola with the lines } y = x \text{ and } y = -x \text{ as axes.}$$

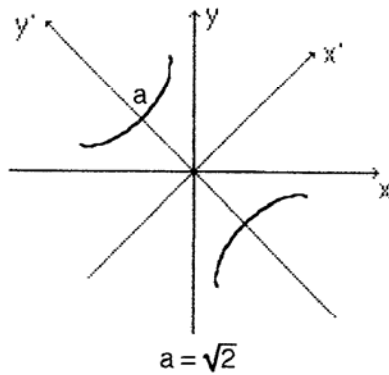


Figure for 2(c)

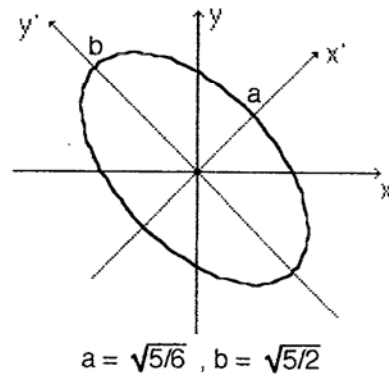


Figure for 2(d)

- (d) $4x^2 + 4xy + 4y^2 - 5 = [x \ y] \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 5$. The symmetric matrix has eigenvalues $\lambda = 6$ and $\lambda = 2$ with corresponding orthonormal eigenvectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ so } C^t \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} C = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}, \text{ where } C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ so}$$

the given equation becomes $[x \ y] C \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} C^t \begin{bmatrix} x \\ y \end{bmatrix} - 5 = 0$, or

$$[x' \ y'] \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - 5 = 0, \text{ where } \begin{bmatrix} x' \\ y' \end{bmatrix} = C^t \begin{bmatrix} x \\ y \end{bmatrix}. \text{ Thus } 6x'^2 + 2y'^2 = 5,$$

$$\text{i.e. } \frac{6x'^2}{5} + \frac{2y'^2}{5} = 1. \text{ The graph is an ellipse with the lines } y = x \text{ and } y = -x \text{ as axes.}$$

3. (a) $a_n = a_{n-1} + 2a_{n-2}$, $a_1 = 1$, and $a_2 = 2$. Let $b_n = a_{n-1}$. Thus $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}$

The matrix has eigenvalues $\lambda = -1$ and $\lambda = 2$ with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \text{ Let } C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}. C^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = C \begin{bmatrix} (-1)^{n-2} & 0 \\ 0 & 2^{n-2} \end{bmatrix} C^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^n + 2^{n-1} \\ 2^{n-1} + 2^{n-2} \end{bmatrix}, \text{ so } a_n = \frac{1}{3} (2^n + 2^{n-1}) = 2^{n-1}.$$

$$a_{10} = 2^9 = 512.$$

- (b) $a_n = 2a_{n-1} + 3a_{n-2}$, $a_1 = 1$, and $a_2 = 3$. Let $b_n = a_{n-1}$. Thus $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}$

The matrix has eigenvalues $\lambda = -1$ and $\lambda = 3$ with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \text{ Let } C = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}. C^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = C \begin{bmatrix} (-1)^{n-2} & 0 \\ 0 & 3^{n-2} \end{bmatrix} C^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3^n + 3^{n-1} \\ 3^{n-1} + 3^{n-2} \end{bmatrix}, \text{ so } a_n = \frac{1}{4} (3^n + 3^{n-1}) = 3^{n-1}.$$

$$a_9 = 3^8 = 6561.$$

8. In the Figure below:
 $x' = OA' = OC + CA' = OC + DP = OB \sin \theta + BP \cos \theta = y \sin \theta + x \cos \theta$
 $y' = OB' = PA' = BC - BD = OB \cos \theta - BP \sin \theta = y \cos \theta - x \sin \theta$

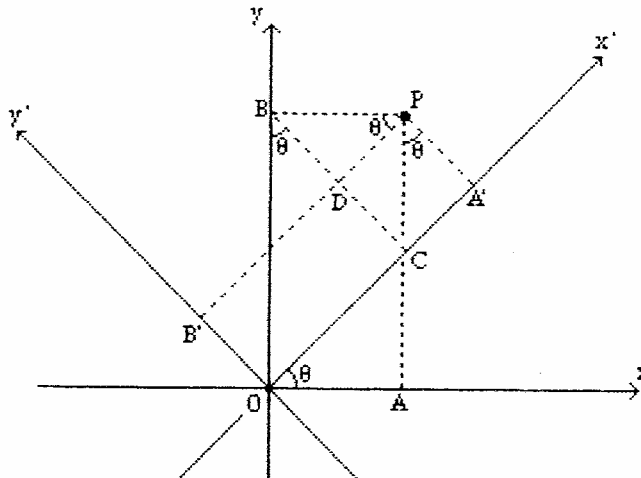


Figure for Exercise 8

4.1

1. $\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ in M_{22} ; k and l are scalars.

axiom 2 $k\mathbf{u} = k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ is in M_{22} .

axiom 7 $k(\mathbf{u} + \mathbf{v}) = k \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} k(a+e) & k(b+f) \\ k(c+g) & k(d+h) \end{bmatrix} = \begin{bmatrix} ka+ke & kb+kf \\ kc+kg & kd+kh \end{bmatrix}$
 $= \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} + \begin{bmatrix} ke & kf \\ kg & kh \end{bmatrix} = k\mathbf{u} + k\mathbf{v}.$

axiom 8 $(k+l)\mathbf{u} = (k+l) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k+l)a & (k+l)b \\ (k+l)c & (k+l)d \end{bmatrix} = \begin{bmatrix} ka+la & kb+lb \\ kc+lc & kd+ld \end{bmatrix}$
 $= \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} + \begin{bmatrix} la & lb \\ lc & ld \end{bmatrix} = k\mathbf{u} + l\mathbf{u}.$

axiom 9 $k(l\mathbf{u}) = k \begin{bmatrix} la & lb \\ lc & ld \end{bmatrix} = \begin{bmatrix} kla & klb \\ klc & kld \end{bmatrix} = \begin{bmatrix} (kl)a & (kl)b \\ (kl)c & (kl)d \end{bmatrix} = (kl) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (kl)\mathbf{u}.$

axiom 10 $1\mathbf{u} = 1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{u}.$

8. Let A and B be 3×3 symmetric matrices. Thus $A=A^t$ and $B=B^t$. $(A+B)^t = A^t + B^t = A+B$. Therefore $A+B$ is symmetric. Closure under addition. Let c be a scalar. $(cA)^t = cA^t = cA$. cA is symmetric. Closure under scalar multiplication. Let $\mathbf{0}$ be the zero 3×3 matrix. $\mathbf{0}^t = \mathbf{0}$. Thus the zero matrix is symmetric. All other properties are satisfied as for 3×3 matrices. The set of 3×3 matrices is a vector space.

11a

- (a) The set of all continuous functions on $[0,1]$ is closed under addition and scalar multiplication – sum of two continuous functions is continuous, and so is the scalar multiple, on $[0,1]$. It is a subset of the set of all functions on $[0,1]$, which is a vector space by the same reasoning as that for the set of all functions with domain the real numbers. It inherits all the other vector space properties from this larger space. Thus the set of all continuous functions on $[0,1]$ is a vector space.

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- (b) $(a, -a, 2a) + (b, -b, 2b) = (a+b, -(a+b), 2(a+b))$ and $c(a, -a, 2a) = (ca, -ca, 2ca)$; thus the sum and scalar product of vectors in the set are also in the set, and so the set is a subspace of $\mathbf{R}^3$. The set is the line defined by the vector $(1, -1, 2)$.
- (d) $(a, b, a-b) + (c, d, c-d) = (a+c, b+d, (a+c)-(b+d))$ and $k(a, b, a-b) = (ka, kb, ka-kb)$; thus the sum and scalar product of vectors in the set are also in the set. This set is all of $\mathbf{R}^3$.

20. (a) Yes, this set is a subspace of $\mathbf{R}^3$. $(a,b,c)+(d,e,f) = (a+d,b+e,c+f)$ and $(a+d)+(b+e)+(c+f) = (a+b+c)+(d+e+f) = 0$. Also $k(a,b,c) = (ka,kb,kc)$ and $ka+kb+kc = k(a+b+c) = 0$.
- (b) No, this set is not a subspace of $\mathbf{R}^3$. Neither the sum nor the scalar product of these vectors is in the set. $(a+b+c)+(d+e+f) = 1+1 = 2$ and $k(a+b+c) = k$.

27.

(b) $2 \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & 10 \end{bmatrix}$, which is not in the set, so the set is not a subspace.

(c) For $k \neq 0$ or 1 , $k \begin{bmatrix} a & a^2 \\ b & b^2 \end{bmatrix} = \begin{bmatrix} ka & ka^2 \\ kb & kb^2 \end{bmatrix} \neq \begin{bmatrix} ka & (ka)^2 \\ kb & (kb)^2 \end{bmatrix}$, so the set is not a subspace.

30. Every element of P_2 is an element of P_3 . Both P_2 and P_3 are vector spaces with the same operations and the same set of scalars, so P_2 is a subspace of P_3 .

4.2

2.

(b) $a(1,-1,0)+b(2,1,4)+c(-2,4,1)=(-2,11,7)$ gives $a+2b-2c=-2$, $-a+b+4c=11$, $4b+c=7$. Unique solution $a=2, b=1, c=3$. Thus $(-2,11,7) = 2(1,-1,0) + (2,1,4) + 3(-2,4,1)$. Is a linear combination.

(e) $a((1,0,1)+b(1,1,0)+c(3,1,2)) = (1,4,-3)$ gives $a+b+3c=1$, $b+c=4$, $a+2c=-3$ has many solutions. The general solution is $a=-3-2c$, $b=4-c$. Thus many linear combinations. $(1,4,-3)=(3-2c)(1,0,1)+(4-c)(1,1,0)+c(3,1,2)$.

5.

(b) $(x_1, x_2, x_3) = a(1,3,1) + b(-1,1,0) + c(4,1,1)$ if and only if $x_1 = a - b + 4c$, $x_2 = 3a + b + c$, $x_3 = a + c$. The system has solution $a = -x_1 - x_2 + 5x_3$, $b = 2x_1 + 3x_2 - 11x_3$, $c = x_1 + x_2 - 4x_3$. Thus the given vectors span $\mathbf{R}^3$. Have $(1,3,-2) = -14(1,3,1) + 33(-1,1,0) + 12(4,1,1)$.

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(b) $\begin{bmatrix} 7 & 6 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} + 4 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.