APMA 308
HW #8
3.4: 27, 31
3.5: 2, 9
5.3: 8c, 9b,16

3.4

27. $|A - 0I_n| = |A|$, so $|A - 0I_n| = 0$ if and only if $|A| = 0$.

31. The characteristic polynomial for the nxn zero matrix O_n is λ^n. Thus the only eigenvalue of O_n is $\lambda = 0$. However if λ is an eigenvalue of A then λ^k is an eigenvalue of $A^k = O_n$ (see Exercise 26). Thus $\lambda^k = 0$ so $\lambda = 0$.

3.5

2. The eigenvectors of $\lambda = 1$ are vectors of the form $r\begin{bmatrix} .7 \\ 1.3 \\ 1 \end{bmatrix}$. If there is no change in total population $.7r + 1.3r + r = 82 + 163 + 52 = 297$, so $r = 297/3$. Thus the long-term prediction is that population in the cities will be $.7r =$ 69.3 million, population in the suburbs will be $1.3r = 128.7$ million, and population in nonmetropolitan areas will be r=99 million.

9. The sum of the terms in each column of a stochastic matrix A is 1, so the sum of the terms in each column of $A - I$ is zero. It has previously been proved (Exercise 18, Section 3.2) that if the sum of the terms in each column of a matrix is zero, the determinant of the matrix is zero. Thus $|A - 1I| = |A - I| = 0$, and 1 is an eigenvalue of A.

5.3

8c.

(c) $\begin{vmatrix} 1-\lambda & 2 & -2 \\ 2 & 4-\lambda & -4 \\ -2 & -4 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)^2 + 24\lambda - 16 = \lambda^2(9-\lambda)$, so the eigenvalues are $\lambda = 0$ and $\lambda = 9$. For $\lambda = 0$, the eigenvectors are $r\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and for $\lambda = 9$, the eigenvectors are $t\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

$$\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2\sqrt{5}/15 & 4\sqrt{5}/1 & \sqrt{5}/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} & 2\sqrt{5}/15 & 1/3 \\ 1/\sqrt{5} & 4\sqrt{5}/15 & 2/3 \\ 0 & \sqrt{5}/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

8c) Gram-Schmidt Orthogonal Process (4.6)

$V_1 = [-2, 1, 0]$ $\quad V_2 = [2, 0, 1]$ $\quad V_3 = [1, 2, -2]$

$u_1 = v_1 = [-2, 1, 0]$

$$u_2 = V_2 - \text{proj}_{u_1} V_2 = V_2 - \frac{(V_2 \cdot u_1)}{(u_1 \cdot u_1)} u_1$$

$$u_3 = V_3 - \text{proj}_{u_1} V_3 - \text{proj}_{u_2} V_3 = V_3 - \frac{(V_3 \cdot u_1)}{(u_1 \cdot u_1)} u_1 - \frac{(V_3 \cdot u_2)}{(u_2 \cdot u_2)} u_2$$

$$u_2 = [2, 0, 1] + \frac{+4}{5}[-2, 1, 0] = [2/5 \quad 4/5 \quad 1]$$

$$u_3 = [1, 2, -2] - \frac{0}{5} u_1 - 0\, u_2 = [1, 2, -2]$$

$u_1 = [-2, 1, 0]$ $\quad \|u_1\| = \sqrt{5}$

$u_2 = [2/5, 4/5, 1]$ $\quad \|u_2\| = \frac{3\sqrt{5}}{5}$

$u_3 = [1, 2, -2]$ $\quad \|u_3\| = 3$

orthonormal basis

$$\left\{\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right), \left(\frac{2\sqrt{5}}{15}, \frac{4\sqrt{5}}{15}, \frac{\sqrt{5}}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)\right\}$$

$$C = \begin{bmatrix} -2/\sqrt{5} & 2\sqrt{5}/15 & 1/3 \\ 1/\sqrt{5} & 4\sqrt{5}/15 & 2/3 \\ 0 & \sqrt{5}/3 & -2/3 \end{bmatrix} \qquad C^{-1} = C^T = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2\sqrt{5}/15 & 4\sqrt{5}/15 & \sqrt{5}/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

9b.

(b) $$\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix}^5 = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}^5 \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 100000 & 0 \\ 0 & 3125 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 80625 & 38750 \\ 38750 & 22500 \end{bmatrix}.$$

Reference 7b. to solve 9b.

7b.

(b) $\begin{vmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = (9-\lambda)(6-\lambda) - 4 = \lambda^2 - 15\lambda + 50 = (\lambda-10)(\lambda-5)$, so the eigenvalues are $\lambda = 10$ and $\lambda = 5$, with corresponding eigenvectors

$r\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $s\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$ is an orthonormal basis.

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

16. Let $D = C^{-1}AC$ be the diagonalization of A. If the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $|D| = \lambda_1 \lambda_2 \ldots \lambda_n$. However $|A| = |D|$, so $|A| = \lambda_1 \lambda_2 \ldots \lambda_n$.