

Section 1.2

$$(b) \begin{bmatrix} 1 & 1 & 0 & 5 & 1 \\ 2 & 3 & 1 & 13 & 2 \end{bmatrix} \xrightarrow{R2+(-2)R1} \begin{bmatrix} 1 & 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{R1+(-1)R2} \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix},$$

so the solutions are in turn $x_1 = -1$, $x_2 = 1$; $x_1 = 2$, $x_2 = 3$; and $x_1 = 1$, $x_2 = 0$.

$$(c) \begin{bmatrix} 1 & -2 & 3 & 6 & -5 & 4 \\ 1 & -1 & 2 & 5 & -3 & 3 \\ 2 & -3 & 6 & 14 & -8 & 9 \end{bmatrix} \xrightarrow{\substack{R2+(-1)R1 \\ R3+(-2)R1}} \begin{bmatrix} 1 & -2 & 3 & 6 & -5 & 4 \\ 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R1+(2)R2 \\ R3+(-1)R2}} \begin{bmatrix} 1 & 0 & 1 & 4 & -1 & 2 \\ 0 & 1 & -1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 3 & 0 & 2 \end{bmatrix} \xrightarrow{\substack{R1+(-1)R3 \\ R2+R3}} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 3 & 0 & 2 \end{bmatrix},$$

so the solutions are in turn $x_1 = 1$, $x_2 = 2$, $x_3 = 3$; $x_1 = -1$, $x_2 = 2$, $x_3 = 0$; and $x_1 = 0$, $x_2 = 1$, $x_3 = 2$.

$$(d) \begin{bmatrix} 1 & 2 & -1 & -1 & 6 & 0 \\ -1 & -1 & 1 & 1 & -4 & -2 \\ 3 & 7 & -1 & -1 & 18 & -4 \end{bmatrix} \xrightarrow{\substack{R2+R1 \\ R3+(-3)R1}} \begin{bmatrix} 1 & 2 & -1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 2 & 2 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{\substack{R1+(-2)R2 \\ R3+(-1)R2}} \begin{bmatrix} 1 & 0 & -5 & -5 & 6 & 8 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & 2 & -2 & -2 \end{bmatrix} \xrightarrow{(1/2)R3} \begin{bmatrix} 1 & 0 & -5 & -5 & 6 & 8 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix},$$

so the solutions are in turn $x_1 = 0$, $x_2 = 0$, $x_3 = 1$; $x_1 = 1$, $x_2 = 2$, $x_3 = -1$; and $x_1 = 3$, $x_2 = -2$, $x_3 = -1$.

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1. (a) Yes. (b) Yes.
- (c) No. The second column contains a leading 1, so other elements in that column should be zero.
- (d) No. The second row does not have 1 as the first nonzero number.
- (e) Yes. (f) Yes. (g) Yes.

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- (h) No. The second row does not have 1 as the first nonzero number. (i) Yes.
2. (a) No. The leading 1 in row 2 is not to the right of the leading 1 in row 3.
 (b) Yes. (c) Yes.
 (d) No. The fourth and fifth columns contain leading 1s, so the other numbers in those columns should be zeros.
 (e) No. The row containing all zeros should be at the bottom of the matrix.
 (f) Yes.
 (g) No. The leading 1 in row 2 is not to the right of the leading 1 in row 3. Also, since column 3 contains a leading 1, all other numbers in that column should be zero.
 (h) No. The leading 1 in row 3 is not to the right of the leading 1s in rows 1 and 2.
 (i) Yes.
3. (a) $x_1 = 2, x_2 = 4, x_3 = -3$. (b) $x_1 = 3r + 4, x_2 = -2r + 8, x_3 = r$.
 (c) $x_1 = -3r + 6, x_2 = r, x_3 = -2$. (d) There is no solution. The last row gives $0 = 1$.
 (e) $x_1 = -5r + 3, x_2 = -6r - 2, x_3 = -2r - 4, x_4 = r$.
 (f) $x_1 = -3r + 2, x_2 = r, x_3 = 4, x_4 = 5$.
4. (a) $x_1 = -2r - 4s + 1, x_2 = 3r - 5s - 6, x_3 = r, x_4 = s$.
 (b) $x_1 = 3r - 2s + 4, x_2 = r, x_3 = s, x_4 = -7$.
 (c) $x_1 = 2r - 3s + 4, x_2 = r, x_3 = -2s + 9, x_4 = s, x_5 = 8$.
 (d) $x_1 = -2r - 3s + 6, x_2 = -5r - 4s + 7, x_3 = r, x_4 = -9s - 3, x_5 = s$.

5. (a)
$$\begin{bmatrix} 1 & 4 & 3 & 1 \\ 2 & 8 & 11 & 7 \\ 1 & 6 & 7 & 3 \end{bmatrix} \xrightarrow[\substack{R2 + (-2)R1 \\ R3 + (-1)R1}]{\sim} \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 0 & 5 & 5 \\ 0 & 2 & 4 & 2 \end{bmatrix} \xrightarrow[\substack{R2 \leftrightarrow R3}]{\sim} \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

many no many
solutions solution solutions

11. (a) If $ax_0 + by_0 = 0$ then $k(ax_0 + by_0) = 0$ so that $a(kx_0) + b(ky_0) = 0$.
Thus $x = kx_0$, $y = ky_0$ is a solution. Likewise for the equation $cx + dy = 0$.
- (b) If $ax_0 + by_0 = 0$ and $ax_1 + by_1 = 0$ then $ax_0 + by_0 + ax_1 + by_1 = 0 + 0 = 0$.
But $ax_0 + by_0 + ax_1 + by_1 = ax_0 + ax_1 + by_0 + by_1 = a(x_0 + x_1) + b(y_0 + y_1)$
so that $a(x_0 + x_1) + b(y_0 + y_1) = 0$. Thus $x = x_0 + x_1$, $y = y_0 + y_1$ is a solution.
Likewise for the equation $cx + dy = 0$.
12. $a(0) + b(0) = 0$ and $c(0) + d(0) = 0$, so $x = 0$, $y = 0$ is a solution.
Multiply 1st equation by c , 2nd by a to eliminate x . Get
 $cax + cby = 0$ and $acx + ady = 0$. Subtract, $ady - cby = 0$, $(ad - bc)y = 0$. Similarly $(ad - bc)x = 0$.
If $ad - bc \neq 0$, $x = 0, y = 0$. If $ad - bc = 0$ the x and y can be anything; thus many solutions.
Therefore $x = 0, y = 0$ is the only solution if and only if $ad - bc \neq 0$.
13. (a) and (b), No. If the first system of equations has a unique solution, then the reduced echelon form of the matrix $[A:B_1]$ will be $[I_3:X]$. The reduced echelon form of $[A:B_2]$ must therefore be $[I_3:Y]$. So the second system must also have a unique solution.
- (c) If the first system of equations has many solutions, then at least one row of the reduced echelon form of $[A:B_1]$ will consist entirely of zeros. Therefore the corresponding row(s) of the reduced echelon form of $[A:B_2]$ will have zeros in the first three columns. If any such row has a nonzero number in the fourth column, the system will have no solution.

14. (a) $\begin{bmatrix} 1 & 1 & 5 & 2 & 3 \\ 1 & 2 & 8 & 5 & 2 \\ 2 & 4 & 16 & 10 & 4 \end{bmatrix} \xrightarrow[R3+(-2)R1]{R2+(-1)R1} \begin{bmatrix} 1 & 1 & 5 & 2 & 3 \\ 0 & 1 & 3 & 3 & -1 \\ 0 & 2 & 6 & 6 & -2 \end{bmatrix} \xrightarrow[R3+(-2)R2]{R1+(-1)R2} \begin{bmatrix} 1 & 0 & 2 & -1 & 4 \\ 0 & 1 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

so the general solution to the first system is $x_1 = -1 - 2r$, $x_2 = 3 - 3r$, $x_3 = r$, and the general solution to the second system is $x_1 = 4 - 2r$, $x_2 = -1 - 3r$, $x_3 = r$.

(b) $\begin{bmatrix} 1 & 2 & 4 & 8 & 5 \\ 1 & 1 & 2 & 5 & 3 \\ 2 & 3 & 6 & 13 & 11 \end{bmatrix} \xrightarrow[R3+(-2)R1]{R2+(-1)R1} \begin{bmatrix} 1 & 2 & 4 & 8 & 5 \\ 0 & -1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 3 & 5 \end{bmatrix} \xrightarrow{(-1)R2} \begin{bmatrix} 1 & 2 & 4 & 8 & 5 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 3 & 5 \end{bmatrix}$

Section 1.3

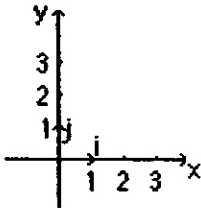
$$\approx \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \text{ so the general solution to the first system is}$$

$x_1 = 2, x_2 = 3 - 2r, x_3 = r$, and the second system has no solution.

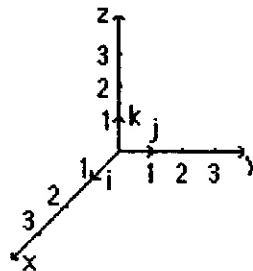
15. A 3×3 matrix represents the equations of three lines in a plane. In order for there to be a unique solution, the three lines would have to meet in a point. For there to be many solutions, the three lines would all have to be the same. It is far more likely that the lines will meet in pairs (or that one pair will be parallel), i.e., that there will be no solution, the situation represented by the reduced echelon form I_3 .
16. A 3×4 matrix represents the equations of three planes. In order for there to be many solutions, the three planes must have at least one line in common. For there to be no solutions, either at least two of the three planes must be parallel or the line of intersection of two of the planes must lie in a plane that is parallel to the third plane. It is more likely that the three planes will meet in a single point, i.e., that there will be a unique solution. The reduced echelon form therefore will be $[I_3 : X]$.
17. The difference between no solution and at least one solution is the presence of a nonzero number in the last position of a row that otherwise consists entirely of zeros. Round-off error is more likely to produce a nonzero number when there should be a zero than the reverse. Thus the answer is (b). Thinking geometrically, a small move by one or more of the linear surfaces (round-off error) may destroy a solution if there is one, but probably won't produce a solution if there is none.

Exercise Set 1.3

1.



2.



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$$(f) \quad 4(-1, 2, 3, -2) = (-4, 8, 12, -8). \quad (g) \quad -5(1, -4, 3, -2, 5) = (-5, 20, -15, 10, -25).$$

$$(h) \quad 3(3, 0, 4, 2, -1) = (9, 0, 12, 6, -3).$$

$$6. \quad (a) \quad \mathbf{u} + \mathbf{w} = (1, 2) + (-3, 5) = (-2, 7). \quad (b) \quad \mathbf{u} + 3\mathbf{v} = (1, 2) + 3(4, -1) = (13, -1).$$

$$(c) \quad \mathbf{v} + \mathbf{w} = (4, -1) + (-3, 5) = (1, 4).$$

$$(d) \quad 2\mathbf{u} + 3\mathbf{v} - \mathbf{w} = 2(1, 2) + 3(4, -1) - (-3, 5) = (17, -4).$$

$$(e) \quad -3\mathbf{u} + 4\mathbf{v} - 2\mathbf{w} = -3(1, 2) + 4(4, -1) - 2(-3, 5) = (19, -20).$$

$$7. \quad (a) \quad \mathbf{u} + \mathbf{w} = (2, 1, 3) + (2, 4, -2) = (4, 5, 1). \quad (b) \quad 2\mathbf{u} + \mathbf{v} = 2(2, 1, 3) + (-1, 3, 2) = (3, 5, 8).$$

$$(c) \quad \mathbf{u} + 3\mathbf{w} = (2, 1, 3) + 3(2, 4, -2) = (8, 13, -3).$$

$$(d) \quad 5\mathbf{u} - 2\mathbf{v} + 6\mathbf{w} = 5(2, 1, 3) - 2(-1, 3, 2) + 6(2, 4, -2) = (24, 23, -1).$$

$$(e) \quad 2\mathbf{u} - 3\mathbf{v} - 4\mathbf{w} = 2(2, 1, 3) - 3(-1, 3, 2) - 4(2, 4, -2) = (-1, -23, 8).$$

$$8. \quad (a) \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (b) \quad 2\mathbf{v} - 3\mathbf{w} = 2 \begin{bmatrix} -1 \\ -4 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -14 \\ 10 \end{bmatrix}.$$

$$(c) \quad 2\mathbf{u} + 4\mathbf{v} - \mathbf{w} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}.$$

$$(d) \quad -3\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = -3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ -4 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 12 \\ -25 \end{bmatrix}.$$

$$9. \quad (a) \quad \mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}. \quad (b) \quad -4\mathbf{v} + 3\mathbf{w} = -4 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -15 \\ 0 \\ 11 \end{bmatrix}.$$

$$(c) \quad 3\mathbf{u} - 2\mathbf{v} + 4\mathbf{w} = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ 15 \end{bmatrix}.$$

$$2\mathbf{u} + 3\mathbf{v} - 8\mathbf{w} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - 8 \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \\ -39 \end{bmatrix}.$$

10. (a) Let W be the subset of vectors of the form $(a, 3a)$. Let $\mathbf{u} = (a, 3a)$, $\mathbf{v} = (b, 3b)$ and k be a scalar. Then $\mathbf{u} + \mathbf{v} = ((a+b), 3(a+b))$ and $k\mathbf{u} = (ka, 3ka)$. The second component of both $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ is 3 times the first component. Thus W is closed under addition and scalar multiplication - it is a subspace.

⑪ b) Let $\underline{u} = (c, -c, d)$ and $\underline{v} = (e, -e, f)$

$$\underline{u} + \underline{v} = (c+e, -c-e, d+f) = (c+e, -(c+e), d+f)$$

We see that $\underline{u} + \underline{v}$ is of the form $(a, -a, b)$
if $a = c+e$ and $b = d+f$. Thus closed under addition

$$k\underline{u} = (kc, -kc, kd)$$

We see that $k\underline{u}$ is of the form $(a, -a, b)$
if $a = kc$ and $b = kd$. Thus closed under multiplication.

$\therefore$ Is a Subspace

c) Let $\underline{u} = (b, 2b, -b)$ and $\underline{v} = (c, 2c, -c)$

$$\underline{u} + \underline{v} = (b+c, 2(b+c), -(b+c)) \text{ which is of the form } (a, 2a, -a), \text{ and so closed under addition}$$

$$k\underline{u} = (kb, 2kb, -kb) \text{ which is of the form } (a, 2a, -a) \text{ and so closed under multiplication.}$$

$\therefore$ Is a subspace

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(b) Let W be the subset of vectors of the form $(a, -a)$. Let $u=(a, -a)$, $v=(b, -b)$ and k be a scalar. Then $u+v=((a+b), -(a+b))$ and $ku=(ka, -ka)$.
The second component of both $u+v$ and ku is minus the first component. Thus W is closed under addition and scalar multiplication - it is a subspace.

(c) Let W be the subset of vectors of the form $(a, 0)$. Let $u=(a, 0)$, $v=(b, 0)$ and k be a scalar. Then $u+v=((a+b), 0)$ and $ku=(ka, 0)$.
The second component of both $u+v$ and ku is zero. Thus W is closed under addition and scalar multiplication - it is a subspace.

(d) Let W be the subset of vectors of the form $(2a, 3a)$. Let $u=(2a, 3a)$, $v=(2b, 3b)$ and k be a scalar. Then $u+v=(2(a+b), 3(a+b))$ and $ku=(2ka, 3ka)$.
The second component of both $u+v$ and ku is $3/2$ times the first component. Thus W is closed under addition and scalar multiplication - it is a subspace.

11. (a) Let W be the subset of vectors of the form (a, b, b) . Let $u=(a, b, b)$, $v=(c, d, d)$ and k be a scalar. Then $u+v=(a+c, b+d, b+d)$ and $ku=(ka, kb, kb)$.
The second and third components of $u+v$ are the same; so are those of ku . Thus W is closed under addition and scalar multiplication - it is a subspace.

(b) Let W be the subset of vectors of the form $(a, -a, b)$. Let $u=(a, -a, b)$, $v=(c, -c, d)$ and k be a scalar. Then $u+v=(a+c, -(a+c), b+d)$ and $ku=(ka, -ka, kb)$.
The second component of $u+v$ minus the first component; and same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace.

(c) Let W be the subset of vectors of the form $(a, 2a, -a)$. Let $u=(a, 2a, -a)$, $v=(b, 2b, -b)$ and k be a scalar. Then $u+v=(a+b, 2(a+b), -(a+b))$ and $ku=(ka, 2kb, -kb)$.
The second component of $u+v$ is twice the first, and the third component is minus the first; and same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace.

(d) Let W be the subset of vectors of the form (a, a, b, b) . Let $u=(a, a, b, b)$, $v=(c, c, d, d)$ and k be a scalar. Then $u+v=(a+c, a+c, b+d, b+d)$ and $ku=(ka, ka, kb, kb)$.
The second and third components of $u+v$ are the same; so are those of ku . Thus W is closed under addition and scalar multiplication - it is a subspace.

12. (a) Let W be the subset of vectors of the form $(a, b, 2a+3b)$. Let $u=(a, b, 2a+3b)$, $v=(c, d, 2c+3d)$ and k be a scalar. Then $u+v=(a+c, b+d, 2(a+c)+3(b+d))$ and $ku=(ka, kb, 2ka+3kb)$. The third component of $u+v$ is twice the first plus three times the second; same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace.

(b) Let W be the subset of vectors of the form $(a, b, 3)$. Let $u=(a, b, 3)$, $v=(c, d, 3)$ and k be a scalar. Then $u+v=(a+c, b+d, 6)$. The third component is not 3. Thus $u+v$ is not in W . W is not a subspace. Let us check closure under scalar multiplication. $ku=(ka, kb, 3k)$. Thus unless $k=1$, ku is not in W . W is not closed under scalar multiplication either.

(c) Let W be the subset of vectors of the form $(a, a+2, b)$. Let $u=(a, a+2, b)$, $v=(c, c+2, d)$ and k be a scalar. Then $u+v=(a+c, a+c+4, b+d)$. The second component is not the first plus 2. Thus $u+v$ is not in W . W is not a subspace. Let us check closure under scalar multiplication. $ku=(ka, ka+2k, kd)$. Thus unless $k=1$ ku is not in W . W is not closed under scalar multiplication either.

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(d) Let W be the subset of vectors of the form $(a, -a, 0)$. Let $u=(a, -a, 0)$, $v=(b, -b, 0)$ and k be a scalar. Then $u+v=(a+b, -(a+b), 0)$ and $ku=(ka, -ka, 0)$. The second component of $u+v$ is minus the first and the last component is zero; same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace.

$$13. \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 7 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ General solution } (2r, -r, r).$$

Let W be the subset of vectors of the form $(2r, -r, r)$. Let $u=(2r, -r, r)$, $v=(2s, -s, s)$ and k be a scalar. Then $u+v=(2(r+s), -(r+s), r+s)$ and $ku=(2kr, -kr, kr)$. The first component of $u+v$ is twice the last component, and the second component is minus the last component; same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector $(2, -1, 1)$.

$$14. \begin{bmatrix} 1 & 1 & -7 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -7 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ General solution } (3r, 4r, r).$$

Let W be the subset of vectors of the form $(3r, 4r, r)$. Let $u=(3r, 4r, r)$, $v=(3s, 4s, s)$ and k be a scalar. Then $u+v=(3(r+s), 4(r+s), r+s)$ and $ku=(3kr, 4kr, kr)$. The first component of $u+v$ is three times the last component, and the second component is four times the last component; same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector $(3, 4, 1)$.

$$15. \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ General solution } (2r, r, 0).$$

Let W be the subset of vectors of the form $(2r, r, 0)$. Let $u=(2r, r, 0)$, $v=(2s, s, 0)$ and k be a scalar. Then $u+v=(2(r+s), r+s, 0)$ and $ku=(2kr, kr, 0)$. The first component of $u+v$ is twice the second component, and the last component is zero; same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector $(2, 1, 0)$.

$$16. \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 0 \\ 3 & 7 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ General solution } (5r, -2r, r).$$

Let W be the subset of vectors of the form $(5r, -2r, r)$. Let $u=(5r, -2r, r)$, $v=(5s, -2s, s)$ and k be a scalar. Then $u+v=(5(r+s), -2(r+s), r+s)$ and $ku=(5kr, -2kr, kr)$. The first component of $u+v$ is five times the last component, and the second component is minus two times the last component; same for ku . Thus W is closed under addition and scalar multiplication - it is a subspace. Line defined by vector $(5, -2, 1)$.

17. General solution is $(2r - 2s, r - 3s, r, s)$. Let $u=(2r - 2s, r - 3s, r, s)$ and $v=(2p - 2q, p - 3q, p, q)$. Then $u+v=(2(r+p) - 2(s+q), (r+p) - 3(s+q), r+p, s+q) = (2(r+p) - 2(s+q), (r+p) - 3(s+q), r+p, s+q)$. The first component is twice the third minus twice the fourth. The second component is the third minus three times the fourth - the required

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form. Thus space is closed under addition.

Let k be a scalar. Then $ku = (2kr - 2ks, kr - 3ks, kr, ks)$. This is in correct form. Thus the set of solutions is a subspace.

18. General solution is $(3r + s, r - 4s, r, s)$. (a) Two specific solutions: $r=1, s=1, u = (4, -5, 1, 1)$; $r=-1, s=-1, u = (-7, -1, 2, -2)$. (b) Other solutions: $u+v = (3, -12, 0, 3)$ and say $-2u = (-8, 10, -2, -2)$, $4v = (-4, -28, -4, 8)$. $(-2u) + (4v) = (-12, -18, -6, 6)$. (c) Are solutions for $r=0, s=3, r=2, s=2, r=4, s=5, r=6, s=8$ respectively. $(-12, -18, -6, 6)$

19. General solution is $(2r - s, -3r - 2s, r, s)$. (a) Two specific solutions: $r=1, s=1, u = (1, -5, 1, 1)$; $r=2, s=1, u = (3, -4, 2, -1)$. (b) Other solutions: $u+v = (5, -9, 3, 0)$ and say $2u = (2, -10, 2, 2)$, $3v = (15, -12, 6, -3)$. $(2u) + (3v) = (17, -22, 8, -1)$. (c) Are solutions for $r=3, s=0, r=2, s=2, r=5, s=3, r=6, s=1$ respectively.

20. (a) $u + (v + w) = (u_1, u_2, \dots, u_n) + ((v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n))$
 $= (u_1, u_2, \dots, u_n) + (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)$
 $= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \dots, u_n + (v_n + w_n))$
 $= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \dots, (u_n + v_n) + w_n)$
 $= ((u_1 + v_1), (u_2 + v_2), \dots, (u_n + v_n)) + (w_1, w_2, \dots, w_n)$
 $= ((u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n)) + (w_1, w_2, \dots, w_n) = (u + v) + w.$

(b) $u + (-u) = (u_1, u_2, \dots, u_n) + (-1)(u_1, u_2, \dots, u_n)$
 $= (u_1, u_2, \dots, u_n) + (-u_1, -u_2, \dots, -u_n) = (u_1 - u_1, u_2 - u_2, \dots, u_n - u_n)$
 $= (0, 0, \dots, 0) = 0.$

(c) $(c+d)u = (c+d)(u_1, u_2, \dots, u_n) = ((c+d)u_1, (c+d)u_2, \dots, (c+d)u_n)$
 $= (cu_1 + du_1, cu_2 + du_2, \dots, cu_n + du_n)$
 $= (cu_1, cu_2, \dots, cu_n) + (du_1, du_2, \dots, du_n)$
 $= c(u_1, u_2, \dots, u_n) + d(u_1, u_2, \dots, u_n) = cu + du.$

(d) $1u = 1(u_1, u_2, \dots, u_n) = (1xu_1, 1xu_2, \dots, 1xu_n) = (u_1, u_2, \dots, u_n) = u.$

Exercise Set 1.4

1. Standard basis for $\mathbb{R}^2$: $\{(1, 0), (0, 1)\}$. (a) Let (a, b) be an arbitrary vector in $\mathbb{R}^2$. We can write $(a, b) = a(1, 0) + b(0, 1)$. Thus vectors $(1, 0)$ and $(0, 1)$ span $\mathbb{R}^2$. (b) Let us examine the identity $p(1, 0) + q(0, 1) = (0, 0)$. This gives $(p, 0) + (0, q) = (0, 0)$, $(p, q) = (0, 0)$. Thus $p=0$ and $q=0$. The vectors are linearly independent.
2. Standard basis for $\mathbb{R}^4$: $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. (a) Let (a, b, c, d) be an arbitrary vector in $\mathbb{R}^4$. We can write $(a, b, c, d) = a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) + d(0, 0, 0, 1)$. Thus vectors in basis span $\mathbb{R}^4$. (b) Let us examine the identity $p(1, 0, 0, 0) + q(0, 1, 0, 0) + r(0, 0, 1, 0) + s(0, 0, 0, 1) = (0, 0, 0, 0)$. This gives $(p, q, r, s) = (0, 0, 0, 0)$. Thus $p=0, q=0, r=0, s=0$. The vectors are linearly independent.