Section 1.1

6.

(e) $2x_1 - 3x_2 + 6x_3 = 4$
$7x_1 - 5x_2 - 2x_3 = 3$
$2x_2 + 4x_3 = 0$

(f) $-2x_2 = 4$
$5x_1 + 7x_2 = -3$
$6x_1 = 8$

13.

(b) $$\begin{bmatrix} 1 & 1 & 0 & 5 & 1 \\ 2 & 3 & 1 & 13 & 2 \end{bmatrix} \underset{R2+(-2)R1}{\approx} \begin{bmatrix} 1 & 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix} \underset{R1+(-1)R2}{\approx} \begin{bmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix},$$

so the solutions are in turn $x_1 = -1,\ x_2 = 1;\ x_1 = 2,\ x_2 = 3;$ and $x_1 = 1,\ x_2 = 0.$

(d) $$\begin{bmatrix} 1 & 2 & -1 & -1 & 6 & 0 \\ -1 & -1 & 1 & 1 & -4 & -2 \\ 3 & 7 & -1 & -1 & 18 & -4 \end{bmatrix} \underset{\substack{R2+R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & 2 & -1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 2 & 2 & 0 & -4 \end{bmatrix}$$

$$\underset{\substack{R1+(-2)R3 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & -5 & -5 & 6 & 8 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & 2 & -2 & -2 \end{bmatrix} \underset{(1/2)R3}{\approx} \begin{bmatrix} 1 & 0 & -5 & -5 & 6 & 8 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix},$$

so the solutions are in turn $x_1 = 0,\ x_2 = 0,\ x_3 = 1;\ x_1 = 1,\ x_2 = 2,\ x_3 = -1;$ and $x_1 = 3,\ x_2 = -2,\ x_3 = -1.$

2. (a) Yes. (b) No. The leading 1 in row 3 is not to the right of the leading 1 in row 2.

(c) No. The leading nonzero number in row 2 is not a 1.

(d) No. There is a row of zeros not at the bottom of the matrix.

4. $$\begin{bmatrix}1 & -1 & -1 & 2\\ 1 & -1 & 1 & 2\\ 3 & -2 & 1 & 5\end{bmatrix} \underset{\substack{R2+(-1)R1\\ R3+(-3)R1}}{\approx} \begin{bmatrix}1 & -1 & -1 & 2\\ 0 & 0 & -2 & 0\\ 0 & 1 & 4 & -1\end{bmatrix} \underset{R2 \Leftrightarrow R3}{\approx} \begin{bmatrix}1 & -1 & -1 & 2\\ 0 & 1 & 4 & -1\\ 0 & 0 & -2 & 0\end{bmatrix}$$

$$\underset{(-1/2)R3}{\approx} \begin{bmatrix}1 & -1 & -1 & 2\\ 0 & 1 & 4 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

(a) $x_1 - x_2 - x_3 = 2$, $x_2 + 4x_3 = -1$, and $x_3 = 0$. Back substituting $x_3 = 0$ into the second equation, $x_2 = -1$, and substituting for x_2 and x_3 in the first equation, $x_1 = 1$.

(b) $$\begin{bmatrix}1 & -1 & -1 & 2\\ 0 & 1 & 4 & -1\\ 0 & 0 & 1 & 0\end{bmatrix} \underset{\substack{R1+(-1)R3\\ R2+(-4)R3}}{\approx} \begin{bmatrix}1 & -1 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix} \underset{R1+R2}{\approx} \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}$$, so again $x_1 = 1$, $x_2 = -1$, $x_3 = 0$.

9. Yes. Consider a 2x3 matrix with no zeros in the first column. Find the echelon form leaving the rows in the original order, then do it switching the order of the rows first.

10. $$\begin{bmatrix}* & * & \cdots & *\\ * & * & \cdots & *\\ : & : & & :\\ * & * & \cdots & *\end{bmatrix} \underset{n\ mults}{\approx} \begin{bmatrix}1 & * & \cdots & *\\ * & * & \cdots & *\\ : & : & & :\\ * & * & \cdots & *\end{bmatrix} \underset{\substack{n(n-1)\ mults\\ n(n-1)\ adds}}{\approx} \begin{bmatrix}1 & * & \cdots & *\\ 0 & * & \cdots & *\\ : & : & & :\\ 0 & * & \cdots & *\end{bmatrix}$$

$$\underset{n-1\ mults}{\approx} \begin{bmatrix}1 & * & \cdots & *\\ 0 & 1 & \cdots & *\\ : & : & & :\\ 0 & * & \cdots & *\end{bmatrix} \underset{\substack{(n-2)(n-1)\ mults\\ (n-2)(n-1)\ adds}}{\approx} \begin{bmatrix}1 & * & \cdots & *\\ 0 & 1 & \cdots & *\\ : & : & & :\\ 0 & 0 & \cdots & *\end{bmatrix} \underset{n-2\ mults}{\approx} \cdots$$

Back substitution

$$\underset{1\,mult}{\approx}\begin{bmatrix}1 & * & \dots & * & * \\ 0 & 1 & \dots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{bmatrix}\underset{\substack{n-1\,mults \\ n-1\,adds}}{\approx}\begin{bmatrix}1 & * & \dots & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{bmatrix}\underset{\substack{n-2\,mults \\ n-2\,adds}}{\approx}\dots\underset{\substack{1\,mult \\ 1\,add}}{\approx}\begin{bmatrix}1 & 0 & \dots & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{bmatrix}$$

So the number of multiplications in Gaussian elimination is

$n + (n-1)n + (n-2) + (n-2)(n-1) + \cdots + 2 + (1)2 + 1 = n^2 + (n-1)^2 + \cdots + 1$

$= n(n+1)(2n+1)/6 = n^3/3 + n^2/2 + n/6$. The number of additions is

$(n-1)n + (n-2)(n-1) + \cdots + 2(3) + 1(2) = n^2 - n + (n-1)^2 - (n-1) + \cdots + 2^2 - 2 + (1-1)$

$= n^2 + (n-1)^2 + \cdots + 2^2 + 1 - [\, n + (n-1) + \cdots + 2 + 1] = n(n+1)(2n+1)/6 - n(n+1)/2$

$= n^3/3 + n^2/2 + n/6 - n^2/2 - n/2 = n^3/3 - n/3$.

Back substitution requires equal numbers of multiplications and additions:

$(n-1) + (n-2) + 2 + 1 = (n-1)n/2 = n^2/2 - n/2$. The total number of multiplications is

therefore $n^3/3 + n^2/2 + n/6 + n^2/2 - n/2 = n^3/3 + n^2 - n/3$ and the total number of additions

is $n^3/3 - n/3 + n^2/2 - n/2 = n^3/3 + n^2/2 - 5n/6$.

11.

	Gauss-Jordan		Gauss	
	mult	add	mult	add
n	$n^3/2 + n^2/2$	$n^3/2 - n/2$	$n^3/3 + n^2 - n/3$	$n^3/3 + n^2/2 - 5n/6$
2	6	3	6	3
3	18	12	17	7
4	40	30	36	26
5	75	60	65	50
6	126	105	106	85
7	196	168	161	133
8	288	252	232	196
9	405	360	321	276
10	550	495	430	375