4.8

① b) The domain is $\mathbb{R}^3$ which has dimension 3.
The range is spanned by the columns of B:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 9 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 9 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ $\{(1,0,0),(0,1,0),(0,0,1)\}$

is a basis for the range, which is seen to be dimension 3. By the rank/nullity theorem, the dimension of the kernel is 0. The transformation is one-to-one.

d) $$\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ A basis for

the range is thus $\{(1,0,0),(0,1,0)\}$, which is 2-dimensional. The domain is $\mathbb{R}^3$, which is 3-dimensional. By the rank/nullity theorem, the dimension of the kernel is 0. The transformation is <u>not</u> one-to-one

2. (b) $|B| = 0$, so the transformation is not one-to-one.

3.

(c) This transformation has no fixed points since there are no solutions to the equation $y = y + 1$.

(d) All points in U are fixed points, since T maps each point to itself.

(e) The set of fixed points is the set of all points for which $(x,y) = (y,x)$. This is the set $\{(r,r)\}$.

(f) The set of fixed points is the set of all points for which $(x,y) = (x+y,x-y)$. This set contains only the zero vector.

(g) If **u** and **v** are fixed points of a linear transformation $T: U \rightarrow U$, then $T(\mathbf{u}) = \mathbf{u}$ and $T(\mathbf{v}) = \mathbf{v}$, so that $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{u} + \mathbf{v}$ and $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{u}$. Thus both **u+v** and c**u** are fixed points of T, and the set of fixed points is therefore a subspace of U.

4. (a) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 7x-3y \\ 5x-2y \end{bmatrix}$. $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$. $A = \begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -2 & 3 \\ -5 & 7 \end{bmatrix}$.

T is invertible. $T^{-1}(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -2 & 3 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x+3y \\ -5x+7y \end{bmatrix}$. $T^{-1}(x, y) = (-2x+3y,-5x+7y)$.

$T^{-1}(2, 3) = (5, 11)$.

(b) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 7x+6y \\ 8x+7y \end{bmatrix}$. $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$. $A = \begin{bmatrix} 7 & 6 \\ 8 & 7 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}$.

T is invertible. $T^{-1}(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7x-6y \\ -8x+7y \end{bmatrix}$. $T^{-1}(x, y) = (7x-6y,-8x+7y)$.

$T^{-1}(2, 3) = (-4,5)$.

(c) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x-y \\ -4x+2y \end{bmatrix}$. $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$, $|A|=0$.

T is singular, T^{-1} does not exist.

(d) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x+4y \\ 0 \end{bmatrix}$. $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. $A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$, $|A|=0$.

T is singular, T^{-1} does not exist.

5. (a) $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+y-z\\-3x+2y-z\\3x-3y+2z\end{bmatrix}$. $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\-3\\3\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\-3\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\-1\\2\end{bmatrix}$. $A=\begin{bmatrix}1&1&-1\\-3&2&-1\\3&-3&2\end{bmatrix}$.

$A^{-1}=\begin{bmatrix}1&1&1\\3&5&4\\3&6&5\end{bmatrix}$. T is invertible. $T^{-1}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}1&1&1\\3&5&4\\3&6&5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+y+z\\3x+5y+4z\\3x+6y+5z\end{bmatrix}$.

$T^{-1}(x, y,z) = (x+y+z,3x+5y+4z,3x+6y+5z)$. $T^{-1}(1,-1,2) = (2,6,7)$.

(b) $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+z\\4x+4y+3z\\-4x-3y-3z\end{bmatrix}$. $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\4\\-4\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\4\\-3\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\\-3\end{bmatrix}$. $A=\begin{bmatrix}1&0&1\\4&4&3\\-4&-3&-3\end{bmatrix}$.

$A^{-1}=\begin{bmatrix}-3&-3&-4\\0&1&1\\4&3&4\end{bmatrix}$. T is invertible. $T^{-1}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}-3&-3&-4\\0&1&1\\4&3&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-3x-3y-4z\\y+z\\4x+3y+4z\end{bmatrix}$.

$T^{-1}(x, y,z) = (-3x-3y-4z,y+z,4x+3y+4z)$. $T^{-1}(1,-1,2) = (-8,1,9)$.

(c) $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+2y+3z\\y-z\\x+3y+2z\end{bmatrix}$. $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\\3\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-1\\2\end{bmatrix}$. $A=\begin{bmatrix}1&2&3\\0&1&-1\\1&3&2\end{bmatrix}$, $|A|=0$.

T is singular, T^{-1} does not exist.

(d) $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}y\\z\\x\end{bmatrix}$. $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix}$. $A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

$A^{-1}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$. T is invertible. $T^{-1}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}z\\x\\y\end{bmatrix}$.

$T^{-1}(x, y,z) = (z, x, y)$. $T^{-1}(1,-1,2) = (2, 1, -1)$.

6. T is one-to-one if and only if ker(T) is the zero vector if and only if dim ker(T) = 0 if and only if dim range(T) = dim domain(T).

7. Let T be a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$. Dim domain(T) = 3 and dim range(T) ≤ 2, so dim ker(T) = dim domain(T) – dim range(T) ≥ 1. Thus ker(T) is not the zero vector and T is not one-to-one.

In general, if dim(U) > dim(V) and T: U $\rightarrow$ V is a linear transformation, then dim ker(T) = dim domain(T) – dim range(T) $\ge$ dim(U) – dim(V) ≥ 1. Thus ker(T) is not the zero vector and T is not one-to-one.

8. Converse Theorem 4.31: Let T: U $\rightarrow$ V be a linear transformation. If T preserves linear independence then it is one-to-one.
Proof: Suppose T(**u**) = T(**w**). Then $\mathbf{0} = T(\mathbf{u}) - T(\mathbf{w}) = T(\mathbf{u} - \mathbf{w})$.
Let $\mathbf{u} - \mathbf{w} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \ldots + a_n \mathbf{u}_n$, where $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for U. Thus $\mathbf{0} = T(\mathbf{u} - \mathbf{w}) = T(a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \ldots + a_n \mathbf{u}_n) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + \ldots + a_n T(\mathbf{u}_n)$.
But $\{T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)\}$ is a linearly independent set in V so $a_1 = a_2 = \ldots = a_n = 0$.
Therefore $\mathbf{u} - \mathbf{w} = \mathbf{0}$, $\mathbf{u} = \mathbf{w}$, and T is one-to-one.

9. (a) $T(u_1)=v_1$ and $T(u_2)=v_2$. $T(u_1)+T(u_2)= v_1+v_2$. $T(u_1+u_2)= v_1+v_2$ since T is linear. Thus $T^{-1}(v_1+v_2)= u_1+u_2$.
(b) $T(u)=v$. $cT(u)=cv$. $T(cu)=cv$, since T is linear. $T^{-1}(cv)=cu$.

10. T_1 and T_2 are invertible. Let standard matrices be A and B. Thus $T_1(u)=Au$, and $T_2(u)=Bu$. $T_2 \circ T_1(u)=T_2(T_1(u))=T_2(A(u))=BA(u)$. Standard matrix of $T_2 \circ T_1$ is BA.
BA is invertible. $(BA)^{-1}=A^{-1}B^{-1}$. Thus $(T_2 \circ T_1)^{-1}(u)= A^{-1}B^{-1}u$ [redacted] $=T_1^{-1} \circ T_2^{-1}(u)$.
[redacted]

<u>Exercise Set 4.9</u>

1. $(r, r, 1) = r(1,1,0) + (0,0,1)$.

2. $(0,1,2) = r(0,0,0) + (0,1,2)$.

3. $(r+1, 2r, r) = r(1,2,1) + (1,0,0)$.

4. $(r-s+3, r, s) = r(1,1,0) + s(-1,0,1) + (3,0,0)$.

4. $(r-s+3, r, s) = (r-s, r, s) + (3, 0, 0)$

5. $(-4r+3s-2, -5r+5s-6, r, s) = r(-4,-5,1,0) + s(3,5,0,1) + (-2,-6,0,0)$.

6. $(-32r-23s-19, 13r+9s+8, -5r-3s-3, r, s)$
[redacted].
$= (-32r-23s, 13r+9s, -5r-3s, r, s) + (-19, 8, -3, 0, 0)$

12. Let $T(\mathbf{x}) = A\mathbf{x}$. First suppose $\ker(T) = \mathbf{0}$. If $A\mathbf{x}_1 = \mathbf{y}$ and $A\mathbf{x}_2 = \mathbf{y}$ then $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$. Thus $\mathbf{x}_1 - \mathbf{x}_2$ is in $\ker(T)$, so $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$; that is, $\mathbf{x}_1 = \mathbf{x}_2$, so there is exactly one solution to $A\mathbf{x} = \mathbf{y}$.

Now suppose $A\mathbf{x} = \mathbf{y}$ has $\mathbf{x}_1$ as its unique solution. If $\mathbf{z}$ is in $\ker(T)$, then $A\mathbf{z} = \mathbf{0}$, so $A(\mathbf{x}_1 + \mathbf{z}) = \mathbf{y} + \mathbf{0} = \mathbf{y}$. Thus $\mathbf{x}_1 + \mathbf{z}$ is a solution to $A\mathbf{x} = \mathbf{y}$. That means $\mathbf{x}_1 + \mathbf{z} = \mathbf{x}_1$, so $\mathbf{z} = \mathbf{0}$, and $\ker(T)$ is the zero vector.

13. (a) $(D^2 + D-2)e^{mx} = 0$, so $(m^2 + m-2)e^{mx} = 0$, which gives $(m+2)(m-1) = 0$, so that $m = -2$ or $m = 1$. Thus a basis for the kernel is the set $\{e^{-2x}, e^x\}$.
$\ker(D^2 + D-2) = \{ae^{-2x} + be^x\}$.

(b) $(D^2 + 4D+3)e^{mx} = 0$, so $(m^2 + 4m+3)e^{mx} = 0$, which gives $(m+3)(m+1) = 0$, so that $m = -3$ or $m = -1$. Thus a basis for the kernel is the set $\{e^{-3x}, e^{-x}\}$.
$\ker(D^2 + 4D+3) = \{ae^{-3x} + be^{-x}\}$.

14. (a) $(D^2 + 5D+6)e^{mx} = 0$ gives $(m^2 + 5m+6)e^{mx} = 0$, which gives $(m+3)(m+2) = 0$, so that $m = -3$ or $m = -2$. Thus a basis for the kernel is the set $\{e^{-3x}, e^{-2x}\}$.

A particular solution is given by $6y = 8$ or $y = 4/3$, so the general solution is $y = re^{-3x} + se^{-2x} + 4/3$.

(b) $(D^2 - 3D)e^{mx} = 0$ gives $(m^2 - 3m)e^{mx} = 0$, which gives $(m-3)m = 0$, so that $m = 3$ or $m = 0$. Thus a basis for the kernel is the set $\{e^{3x}, 1\}$.

A particular solution is given by $-3dy/dx = 8$ or $y = -8x/3$, so the general solution is $y = re^{3x} + s - 8x/3$.

(c) $(D^2 - 7D+12)e^{mx} = 0$ gives $(m^2 - 7m+12)e^{mx} = 0$, which gives $(m-3)(m-4) = 0$, so that $m = 3$ or $m = 4$. Thus a basis for the kernel is the set $\{e^{3x}, e^{4x}\}$.
A particular solution is given by $12y = 24$ or $y = 2$, so the general solution is $y = re^{3x} + se^{4x} + 2$.

(d) $(D^2 + 8D)e^{mx} = 0$ gives $(m^2 + 8m)e^{mx} = 0$, which gives $(m+8)m = 0$, so that $m = -8$ or $m = 0$. Thus a basis for the kernel is the set $\{e^{-8x}, 1\}$.

A particular solution is of the form $y = ke^{2x}$. We determine k. $dy/dx = 2ke^{2x}$ and $d^2y/dx^2 = 4ke^{2x}$, so that $d^2y/dx^2 + 8dy/dx = 20ke^{2x} = 3e^{2x}$. Thus $k = 3/20$, and the general solution is $y = re^{-8x} + s + 3e^{2x}/20$.