Homework 11

SECTION 4.6 3.

(b) $\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\cdot\left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = 0$, so the vectors are orthogonal.

$\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\cdot\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) = 1$ and $\left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\cdot\left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = 1$, ✓

so the vectors are unit vectors. Thus the vectors are an orthonormal set.

(c) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) = 0$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot(0,1,0) = 0$, and

$\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\cdot(0,1,0) = 0$, so the set of vectors is orthogonal. ✓

$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = 1$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) = 1$, and

$(0,1,0)\cdot(0,1,0) = 1$, so the vectors are unit vectors. Thus the vectors are an orthonormal set.

6.

(e) $\mathbf{a}_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. $\|\mathbf{a}_1\|^2 = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$,

$\|\mathbf{a}_2\|^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$, and $\|\mathbf{a}_3\|^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$, so $\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \|\mathbf{a}_3\| = 1$,

and $\mathbf{a}_1\cdot\mathbf{a}_2 = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$, $\mathbf{a}_1\cdot\mathbf{a}_3 = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$, $\mathbf{a}_2\cdot\mathbf{a}_3 = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$.

16. (a) $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (2,0,1) - \dfrac{(2,0,1)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = (1,-1,0)$,

$$\mathbf{u}_3 = (2,4,5) - \frac{(2,4,5)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(2,4,5)\cdot(1,-1,0)}{(1,-1,0)\cdot(1,-1,0)}(1,-1,0)$$

$= (2,4,5) - \dfrac{11}{3}(1,1,1) - (-1)(1,-1,0) = \left(\dfrac{-2}{3},\dfrac{-2}{3},\dfrac{4}{3}\right)$, is an orthogonal basis

for $\mathbf{R}^3$. $\|\mathbf{u}_1\| = \sqrt{3}$, $\|\mathbf{u}_2\| = \sqrt{2}$, and $\|\mathbf{u}_3\| = \dfrac{\sqrt{24}}{3} = \dfrac{2\sqrt{6}}{3}$, so the set

$\{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}),(\frac{1}{\sqrt{2}},\frac{-1}{2},0)(\frac{-1}{\sqrt{6}},\frac{-1}{\sqrt{6}},\frac{2}{\sqrt{6}})\}$ is an orthonormal basis for $\mathbf{R}^3$.

(b) $\mathbf{u}_1 = (3,2,0)$, $\mathbf{u}_2 = (1,5,-1) - \dfrac{(1,5,-1)\cdot(3,2,0)}{(3,2,0)\cdot(3,2,0)}(3,2,0) = (-2,3,-1)$,

$$\mathbf{u}_3 = (5,-1,2) - \frac{(5,-1,2)\cdot(3,2,0)}{(3,2,0)\cdot(3,2,0)}(3,2,0) - \frac{(5,-1,2)\cdot(-2,3,-1)}{(-2,3,-1)\cdot(-2,3,-1)}(-2,3,-1)$$

$= (5,-1,2) - (3,2,0) - \dfrac{-15}{14}(-2,3,-1) = \left(\dfrac{-2}{14},\dfrac{3}{14},\dfrac{13}{14}\right)$, is an orthogonal basis

for $\mathbf{R}^3$. $\|\mathbf{u}_1\| = \sqrt{13}$, $\|\mathbf{u}_2\| = \sqrt{14}$, and $\|\mathbf{u}_3\| = \dfrac{\sqrt{182}}{14}$, so the set

$\{(\frac{3}{\sqrt{13}},\frac{2}{\sqrt{13}},0),(\frac{-2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}}),(\frac{-2}{\sqrt{182}},\frac{3}{\sqrt{182}},\frac{13}{\sqrt{182}})\}$

is an orthonormal basis for $\mathbf{R}^3$.

21. First find an orthonormal basis for W. $\mathbf{u}_1 = (0,-1,3)$,

$\mathbf{u}_2 = (1,1,2) - \dfrac{(1,1,2)\cdot(0,-1,3)}{(0,-1,3)\cdot(0,-1,3)}(0,-1,3) = (1,1,2) - \dfrac{1}{2}(0,-1,3) = \left(1,\ \dfrac{3}{2},\dfrac{1}{2}\right)$, is an

orthogonal basis for the subspace of $\mathbf{R}^3$. $\|\mathbf{u}_1\| = \sqrt{10}$ and $\|\mathbf{u}_2\| = \frac{1}{2}\sqrt{14}$, so the set

$\left\{(0,\frac{-1}{\sqrt{10}},\frac{3}{\sqrt{10}}),(\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}})\right\}$ is an orthonormal basis for W.

(b) $\text{proj}_W(1,1,1) = \left((1,1,1)\cdot(0, \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}})\right)\left(0, \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

$+ \left((1,1,1)\cdot(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}})\right)\left(\cancel{\frac{1}{\sqrt{14}}}\,\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) = \frac{1}{5}(0,-1,3) + \frac{3}{7}(2,3,1)$

$= \left(\frac{6}{7}, \frac{38}{35}, \frac{36}{35}\right)$ ✓

24. $V = \{a(1,2,0) + b(0,0,1)\}$. The set $\{(1,2,0), (0,0,1)\}$ is an orthogonal basis. The vectors $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ and $(0,0,1)$ are therefore an orthonormal basis. $\mathbf{v} = \mathbf{w} + \mathbf{w}_\perp$, where

$\mathbf{w} = \text{proj}_V\, \mathbf{v} = \left((4,1,-2)\cdot(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) + \left((4,1,-2)\cdot(0,0,1)\right)(0,0,1)$

$= \frac{6}{5}(1,2,0) + (-2)(0,0,1) = (\frac{6}{5}, \frac{12}{5}, -2)$ and

$\mathbf{w}_\perp = \mathbf{v} - \text{proj}_V\, \mathbf{v} = (4,1,-2) - (\frac{6}{5}, \frac{12}{5}, -2) = \cancel{(\frac{14}{25}, \frac{-7}{25}, 0)}$ $\left(\frac{14}{5}, -\frac{7}{5}, 0\right)$

27. $W = \{a(1,-2,0) + b(0,0,1)\}$. The set $\{(1,-2,0), (0,0,1)\}$ is an orthogonal basis. The vectors $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$ and $(0,0,1)$ are therefore an orthonormal basis.

$\text{proj}_W\, \mathbf{x} = \left((2,4,-1)\cdot(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0)\right)\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right) + \left((2,4,-1)\cdot(0,0,1)\right)(0,0,1)$

$= \frac{-6}{5}(1,-2,0) + (-1)(0,0,1) = (\frac{-6}{5}, \frac{12}{5}, -1)$, so

$d(\mathbf{x}, W) = \|\mathbf{x} - \text{proj}_W\, \mathbf{x}\| = \left\|(2,4,-1) - (\frac{-6}{5}, \frac{12}{5}, -1)\right\| = \left\|(\frac{16}{5}, \frac{8}{5}, 0)\right\| = \frac{8}{5}\sqrt{5}$.

Section 4.7

12. (a) If $\ker(T) = \mathbf{0}$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are a basis for U, then $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)$ are a basis for range(T) ~~(see Exercise 14(a))~~. Thus $\{T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)\}$ is a set of n linearly independent vectors in V. If $\dim(V) = \dim(U) = n$, this set of vectors is a basis for V. Since V and range(T) have the same basis vectors, they are the same vector space.

If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are a basis for U, then $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)$ span range(T). If range(T) = V then dim range(T) = n and $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)$ are a basis for range(T) = V. If $\mathbf{0} = T(\mathbf{u}) = T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n)$ $= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + \ldots + a_nT(\mathbf{u}_n)$ then $a_1 = a_2 = \ldots = a_n = 0$ since $T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)$ are a basis. Thus $\mathbf{u} = \mathbf{0}$ and $\ker(T) = \mathbf{0}$.

(b) If range(T) = $\mathbf{0}$, then $T(\mathbf{u}) = \mathbf{0}$ for all vectors $\mathbf{u}$ in U. Thus ker(T) = U.

If ker(T) = U, then $T(\mathbf{u}) = \mathbf{0}$ for all vectors $\mathbf{u}$ in U. Thus the only vector in range(T) is $\mathbf{0}$.

16. $T(x,y) = (x-y, 2y-2x) = (2,-4)$ if $x-y = 2$ and $2y-2x = -4$, i.e., if $y = x-2$. Thus the set of vectors mapped by T into $(2,-4)$ is the set $\{(r, r-2)\}$.

Section 6.4

16. The system of equations is given by $A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 9 \end{bmatrix}$, with A given in Exercise 15. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

$$\text{Pinv}(A)\begin{bmatrix} 1 \\ 3 \\ 5 \\ 9 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 & 5 & 0 & -5 \\ -3 & -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \\ 9 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -20 \\ 26 \end{bmatrix}, \text{ so}$$

$a = -2$, $b = 13/5$, and the least squares line is $y = -2 + 13x/5$. ✓

22. The system of equations is given by $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 5 \end{bmatrix}$ with A given in Exercise 21. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$

$$\text{Pinv}(A)\begin{bmatrix} 4 \\ 0 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 45 & -15 & -25 & 15 \\ -31 & 23 & 27 & -19 \\ 5 & -5 & -5 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 180 \\ -138 \\ 30 \end{bmatrix}, \text{ so}$$

$a = 9$, $b = -69/10$, $c = 3/2$, and the least squares parabola is $y = 9 - 69x/10 + 3x^2/2$. ✓

27. Let line be N = a + bX. The data points give 101 = a + 20b, 115 = a + 25b, 92 = a + 30b, 64 = a + 35b, 60 = a + 40b, 50 = a + 45b, and 49 = a + 50b.

Thus the system of equations is given by $A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 101 \\ 115 \\ 92 \\ 64 \\ 60 \\ 50 \\ 49 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 20 \\ 1 & 25 \\ 1 & 30 \\ 1 & 35 \\ 1 & 40 \\ 1 & 45 \\ 1 & 50 \end{bmatrix}$.

$$\text{pinv } A = (A^t A)^{-1} A^t = \frac{1}{4900}\begin{bmatrix} 9275 & -245 \\ -245 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 25 & 30 & 35 & 40 & 45 & 50 \end{bmatrix}$$

$$= \frac{1}{4900}\begin{bmatrix} 4375 & 3150 & 1925 & 700 & -525 & -1750 & -2975 \\ -105 & -70 & -35 & 0 & 35 & 70 & 105 \end{bmatrix}.$$ ✓

$\text{pinv}(A)\begin{bmatrix} 101 \\ 115 \\ 92 \\ 64 \\ 60 \\ 50 \\ 49 \end{bmatrix} = \frac{1}{4900}\begin{bmatrix} 761250 \\ -11130 \end{bmatrix}$, so $a = \frac{76125}{490}$, $b = \frac{-1113}{490}$, and the least squares equation is $N = \frac{76125}{490} - \frac{1113}{490}X$. The predicted number of driver fatalities at age 22 is $\frac{76125 - 1113 \times 22}{490} \doteq 105$.