4.3

5. If (0,0,0,0) = a(1,0,0,7) + b(0,1,0,4) + c(0,0,1,3) = (a, b, c, 7a+4b+3c), then equating the first three components of (a, b, c, 7a+4b+3c) and (0,0,0,0) we see that a = 0, b = 0, and c = 0. Thus the vectors are linearly independent.

The set of nonzero row vectors of any matrix in reduced echelon form is linearly independent using the same reasoning as above.

13. Let V = {$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$} be linearly independent. Let {$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$} be a subset. Consider the identity $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m = \mathbf{0}$. This means that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + \ldots + 0\mathbf{v}_n = \mathbf{0}$. Since {$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$} is linearly independent, $a_1=0, \ldots, a_m=0$. Thus {$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$} is linearly independent.

A linearly dependent set can have a linearly independent subset. For example, the set {(1,−2,3), (−1,2,−3), (1,0,0)} is linearly dependent since (1,−2,3) + (−1,2,−3) + 0(1,0,0) = **0**, but the subset {(−1,2,−3), (1,0,0)} is linearly independent.

14. Given $\mathbf{u}\cdot\mathbf{v} = 0$. Consider $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$. Take the dot product with **u** and simplify. $(a\mathbf{u} + b\mathbf{v})\cdot\mathbf{u} = \mathbf{0}\cdot\mathbf{u}$, $a\mathbf{u}\cdot\mathbf{u} + b\mathbf{v}\cdot\mathbf{u} = 0$, $a\mathbf{u}\cdot\mathbf{u} + b\mathbf{u}\cdot\mathbf{v} = 0$, $a\mathbf{u}\cdot\mathbf{u} + 0 = 0$, $a\mathbf{u}\cdot\mathbf{u} = 0$. Since u is a nonzero vector this means that a=0. Similarly on taking the dot product with **v**, b=0. Thus **u** and **v** are linearly independent.

Exercise Set 4.4

1. For two vectors to be linearly dependent one must be a multiple of the other. In each case below, neither vector is a multiple of the other, so the vectors are linearly independent. We show that each set spans $\mathbf{R}^2$:

4.4 1 b) $a(-1,4) + b(2,5) = (-a+2b, 4a+5b)$

$$-a+2b = x$$
$$4a+5b = y$$

Solve for a & b:

$$\begin{bmatrix} -1 & 2 & \vdots & x \\ 4 & 5 & \vdots & y \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & \vdots & -x \\ 4 & 5 & \vdots & y \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & \vdots & -x \\ 0 & 13 & \vdots & -4x+y \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & \vdots & -x \\ 0 & 1 & \vdots & \frac{-4x+y}{13} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & \vdots & \frac{-5x+2y}{13} \\ 0 & 1 & \vdots & \frac{-4x+y}{13} \end{bmatrix} \quad \therefore\ a = \frac{-5x+2y}{13}, \quad b = \frac{-4x+y}{13}$$

$$(x,y) = \frac{-5x+2y}{13}(-1,4) + \frac{-4x+y}{13}(2,5)$$

2. It is necessary to show either that the set of vectors is linearly independent or that the set spans $\mathbf{R}^2$. For two vectors to be linearly dependent, one must be a multiple of the other. In each case neither vector is a multiple of the other, so the vectors are linearly independent and therefore a basis for $\mathbf{R}^2$.

4.4 (4) b) $a(1,2,3)+b(2,4,1)+c(3,0,0)=(0,0,0)$

$$\left[\begin{array}{ccc|c}1&2&3&0\\2&4&0&0\\3&1&0&0\end{array}\right]\approx\left[\begin{array}{ccc|c}1&2&3&0\\0&0&-6&0\\0&-5&-9&0\end{array}\right]\approx\left[\begin{array}{ccc|c}1&2&3&0\\0&-5&-9&0\\0&0&-6&0\end{array}\right]\approx\left[\begin{array}{ccc|c}1&2&3&0\\0&1&9/5&0\\0&0&1&0\end{array}\right]$$

$$\approx\left[\begin{array}{ccc|c}1&0&-3/5&0\\0&1&9/5&0\\0&0&1&0\end{array}\right]\approx\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]\Rightarrow a=b=c=0 \quad \therefore L.I.$$

Since we have 3 L.I. vectors in $\mathbb{R}^3$, by Theorem 4.11a, the set is a basis.

(7) $a(-1,2,1)+b(2,-1,0)+c(1,4,3)=(0,0,0)$

$$\left[\begin{array}{ccc|c}-1&2&1&0\\2&-1&4&0\\1&0&3&0\end{array}\right]\approx\left[\begin{array}{ccc|c}1&-2&-1&0\\2&-1&4&0\\1&0&3&0\end{array}\right]\approx\left[\begin{array}{cccc}1&-2&-1&0\\0&3&6&0\\0&2&4&0\end{array}\right]$$

$$\approx\left[\begin{array}{cccc}1&-2&-1&0\\0&1&2&0\\0&2&4&0\end{array}\right]\approx\left[\begin{array}{ccc|c}1&0&3&0\\0&1&2&0\\0&0&0&0\end{array}\right]\Rightarrow a=-3c,\ b=-2c \quad \therefore L.D.$$

$-3(-1,2,1)-2(2,-1,0)+(1,4,3)=0 \Rightarrow (1,4,3)=3(-1,2,1)+2(2,-1,0)$

$\therefore$ $(1,4,3)$ is redundant and may be removed.

Since $(-1,2,1)$ is not a scalar multiple of $(2,-1,0)$, they are L.I. The set $\{(-1,2,1),(2,-1,0)\}$ is therefore a basis for the subspace generated by $\{(-1,2,1),(2,-1,0),(1,4,3)\}$, which is seen to be two-dimensional.

15. (a) (a,a,b) = a(1,1,0) + b(0,0,1), so the linearly independent set {(1,1,0),(0,0,1)} spans the subspace of vectors of the form (a,a,b) and is therefore a basis. The dimension of the space is 2 since there are 2 vectors in the basis.

(b) (a,a,2a) = a(1,1,2), so the single vector (1,1,2) is a basis for the subspace of vectors of the form (a,a,2a). The dimension of the space is 1 since there is 1 vector in the basis.

(c) (a, b, a+b) = a(1,0,1) + b(0,1,1), so the linearly independent set {(1,0,1),(0,1,1)} spans the subspace of vectors of the form (a,b,a+b) and is therefore a basis. The dimension of the space is 2 since there are 2 vectors in the basis.

(d) (a, 2b, a+3b) = a(1,0,1) + b(0,2,3), so the linearly independent set {(1,0,1),(0,2,3)} spans the subspace of vectors of the form (a,2b,a+3b) and is therefore a basis. The dimension of the space is 2 since there are 2 vectors in the basis.

17. (a) The set $\{x^3, x^2, x, 1\}$ is a basis. The dimension is 4.

(b) $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},$
$\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}$, and $\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$ are a basis. The dimension is 9.

(c) $\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}$, and $\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}$

are a basis. The dimension is 6.

4.5

2 (f) R3=2R2-R1. The third row is in the space spanned by the other two linearly independent rows. Dim row space = 2. Rank = 2.

4. (b) $\begin{bmatrix}1&2&0\\0&1&1\\-1&2&3\end{bmatrix} \approx \begin{bmatrix}1&2&0\\0&1&1\\0&4&3\end{bmatrix} \approx \begin{bmatrix}1&0&-2\\0&1&1\\0&0&-1\end{bmatrix} \approx \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, so the vectors (1,0,0),

(0,1,0), and (0,0,1) are a basis for the row space and the rank of the matrix is 3.

6. (b) ~~The given vectors are linearly independent so they are a basis for the space they~~

~~span, which is R^3. Also~~ $\begin{bmatrix}1&-2&5\\1&-1&4\\2&-5&14\end{bmatrix} \approx \begin{bmatrix}1&-2&5\\0&1&-1\\0&-1&4\end{bmatrix} \approx \begin{bmatrix}1&0&3\\0&1&-1\\0&0&1\end{bmatrix} \approx \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

The three vectors are L.I., and
so the row vectors of any of these matrices are a basis for the [redacted] vector space.

9. $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 2 & 5 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so the vectors (1,0,5) and (0,1,−1)

are a basis for the row space of A.

$A^t = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 2 & 1 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so the vectors (1,0,3) and (0,1,−1)

are a basis for the row space of A^t. Therefore the column vectors

$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are a basis for the column space of A. Both the row space and the

column space of A have dimension 2.

10 (b) (i) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. (ii) Ranks 2 and 2. Many solutions. (iii) $x_1 = -3r + 4$, $x_2 = -2r + 1$, $x_3 = r$.

(iv) $\begin{bmatrix} 9 \\ 7 \\ 7 \end{bmatrix} = (-3r + 4)\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + (-2r + 1)\begin{bmatrix} -3 \\ -1 \\ -5 \end{bmatrix} + r\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$.

(c) (i) $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. (ii) Ranks 2 and 3. No solution. (iii) No solution.

(iv) $\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

11. Use Theorem 4.17. (a) Unique solution. (b) No solution. (c) Many solutions. (d) No solution. (e) Many solutions. (f) Unique solution.

19. Use Theorem 4.18 and an awareness of Theorem 4.11.
(a)⇔(b) by Thm 4.18 (a) & (d). (a)⇔(f) by Thm 4.18 (a) & (c) One set {(a),(b),(f)}.
(c)⇔(d) by Thm 4.18 (e) & (b). (d)⇔(e) by Thm 4.18 (b) & (e). Other set {(c),(d),(e)}.