Name: Solutions ☐ TTH ☐ MWF 10AM ☐ MWF 11AM

Pledge: ______________________________

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Signature: ______________________________

To get credit for a problem, you must show all of your reasoning and calculations. No calculators may be used.

Problem	**Score**
Multiple Choice	
1	
2	
3	
4	
5	
6	
Total	

Multiple Choice Answers

1	A
2	D
3	D
4	C
5	A
6	B
7	E
8	C
9	A
10	C
11	B
12	D
13	E
14	A
15	B
16	C

Multiple Choice (5 Points Each)

1. Find the dot product of $\mathbf{u} = [1,3,7]$ and $\mathbf{v} = [-6,2,1]$.

   (a) 7 (b) 0 (c) -7 (d) 1 (e) 4

2. If A is an invertible $n \times n$ matrix, then which of the following are true?

   i The columns are linearly independent.

   ii The equation $A\mathbf{x} = \mathbf{b}$ might not have a solution for some $\mathbf{b}$.

   iii The rank of A is n.

   (a) i only (b) ii & iii only (c) i & ii only (d) i & iii only (e) all are true.

3. Let R be the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   The system $R\mathbf{x} = \mathbf{0}$ has

   (a) 1 solution (b) 0 solutions (c) 2 solutions (d) infinitely many solutions (e) not enough information.

4. If A is a 2×4 matrix of rank 2, then the dimension of the kernel of $L(\mathbf{v}) = A\mathbf{v}$ is

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

5. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 3 & 2 \end{bmatrix},$$

   then $B - A =$

   (a) $\begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

6. If A is a 3×3 matrix with $|A| = 4$, and if B is the result of scaling the first row of A by 3, then $|B| =$

   (a) 4 (b) 12 (c) -12 (d) 0 (e) Not enough information.

7. The eigenvalues of $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ are

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 - 25$$

   (a) 3, 4 (b) 0, 7 (c) ± 3 (d) ± 4 (e) ± 5

8. Find all values of k such that the set

$$\{[1,1,1],[2,k,4],[3,6,k]\}$$

does *not* form a basis for $\mathbb{R}^3$. $k =$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & k & 6 \\ 1 & 4 & k \end{vmatrix} = k^2+12+12-3k-24-2k = k^2-5k = 0 \Leftrightarrow k=0,5$$

(a) 5 (b) -5 (c) 0, 5 (d) 2 (e) It's always a basis.

For the next two problems, let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 4 & 3 & 3 \\ 3 & 6 & 4 & 4 \end{bmatrix}.$$

$$A^t = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} -2r \\ r \\ -s \\ s \end{bmatrix}$$

9. A basis for the kernel of the map $L(\mathbf{v}) = A\mathbf{v}$ is given by

(a) $\{[-2,1,0,0],[0,0,-1,1]\}$ (b) $\{[-2r,r,s,-s]\}$ (c) $\{[1,2,0,0],[0,0,1,1]\}$
(d) $\{[1,1,2,-1],[1,2,3,2],[2,2,4,2]\}$ (e) $\{[0,0,1,-1]\}$

10. A basis for the column space of A is given by

(a) $\{[-1,-1,1,0],[-2,-1,0,1]\}$ (b) $\{[-r-2s,-r-s,r,s]\}$ (c) $\{[1,0,1,2],[0,1,1,1]\}$
(d) $\{[1,1,2,-1],[1,2,3,2],[2,2,4,2]\}$ (e) $\{[0,0,1,-1]\}$

11. Let $f(x) = x^2$. Then the second order Fourier approximation to f over the interval $[-\pi,\pi]$ is

f even $\Rightarrow$ no sin terms. $\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^3}{3}$. $\int_{-\pi}^{\pi} x^2 \cos mx\,dx$

(a) $2\sin x - \sin 2x$ (b) $\frac{\pi^2}{3} + 4\cos x - \cos 2x$ (c) $1 + \cos x + \cos 2x$
(d) $\frac{4\pi^3}{3} + 4\pi\cos x - \pi\cos 2x$ (e) Cannot be determined.

$x^2 \quad \cos mx$
$2x \quad \frac{1}{m}\sin mx$
$2 \quad -\frac{1}{m^2}\cos mx$
$0 \quad -\frac{1}{m^3}\sin mx$

$= \frac{x^2}{m}\sin mx + \frac{2x}{m^2}\cos mx - \frac{2}{m^3}\sin mx \Big| = \frac{4\pi}{m^2}(-1)^m$

12. If $p(x) = x^2 + 3x + 5 \in P_2(x)$, and if $\mathcal{B} = \{x+1, 2, x^2+x+1\}$, then $[p]_{\mathcal{B}} =$

(a) $[1,3,5]$ (b) $[2,2,1]$ (c) $[2,1,2]$ (d) $[2,1,1]$ (e) $[1,2,1]$

$p(x) = 2(x+1) + 1\cdot 2 + 1(x^2+x+1)$

13. Which of the following are subspaces of $P_3(x)$?

i $\{p(x) | p''(x) + p'(x) + p(x) = 0\}$ ✓
ii $\{p(x) | p(3) = 3\}$ ✗
iii $\{p(x) | p(4) = 0\}$ ✓
iv $\{p(x) | p'(x) = p(0)\}$ ✓

(a) i only (b) iii only (c) i & iv only (d) i, ii, & iii only (e) i, iii, & iv only.

14. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$, then the adjoint of A is $adj(A) =$ $\begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix}^t$

(a) $\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$

15. If $\mathbf{v}_1 = x^3 + x^2 + x$, $\mathbf{v}_2 = x^2 + x + 1$, and $\mathbf{v}_3 = x^3 + x + 1$, then express $3x^2 + 2x + 1$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, & $\mathbf{v}_3$: $\bar{v}_1 + 2\bar{v}_2 - \bar{v}_3$

(a) $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ (b) $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$ (c) $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$ (d) $-\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3$ (e) Not in the span.

16. Let $Q(x, y) = 4x^2 + 6xy - 4y^2$. Then the graph of $Q(x, y) = 1$ is an

(a) ellipse (b) parabola (c) hyperbola (d) line (e) None of these.

$$Q(x,y) = \bar{x}^t \underbrace{\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}}_{A} \bar{x}$$

$\det(A - \lambda I) = \lambda^2 - 25$

$\Rightarrow$ eigenval: $\lambda = \pm 5 \Rightarrow$ hyperbola

1. (a) (12 Points)Let

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & -2 & 3 \\ 4 & 8 & a \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ c \end{bmatrix}.$$

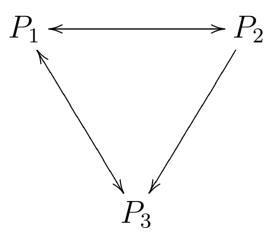
For what values of a and c does the system $A\mathbf{x} = \mathbf{b}$ have

(i) Many solutions?

(ii) A unique solution?

(iii) No solutions?

(b) (8 Points) Find the adjacency matrix A of the following communication network. Find the number of 2-paths from P_1 to P_2, from P_1 to P_3, and from P_2 to P_3 using the matrix A^2



$$\begin{array}{c} -2 \\ -4 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & 6 & 5 \\ 1 & -2 & 3 & 7 \\ 4 & 8 & a & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 8 & 0 & -9 \\ 0 & 16 & a-12 & c-28 \end{array}\right] -2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 8 & 0 & -9 \\ 0 & 0 & a-12 & c-10 \end{array}\right]$$

$a \neq 12 \Rightarrow$ unique solution

$a = 12$: $c \neq 10 \Rightarrow$ no solution

$c = 10 \rightarrow$ many solutions

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = A$$

$$A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

So 0 2-stage paths $P_1 \to P_2$

1 2-stage path $P_1 \to P_3$

1 2-stage path $P_2 \to P_3$

2. (a) (6 Points) Find an LU decomposition (if possible) of the matrix

$$C = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -1 & 5 & 9 \end{bmatrix}$$

(b) (10 Points) Solve the system $A\mathbf{x} = \mathbf{b}$ using the LU factorization $A = LU$;

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

(c) (4 Points) Using Cramer's rule solve the following system

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ -x_1 + x_2 &= 5 \end{aligned}$$

$$+1 \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -1 & 5 & 9 \end{bmatrix} -2 \implies \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 7 & 6 \end{bmatrix} + 7/3 \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 0 & 20 \end{bmatrix}$$

$$L: \quad \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & ? & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -7/3 & 1 \end{bmatrix}$$

$L\bar{y} = \bar{b}$

$$\begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$x = 2$

$\frac{1}{2}x + y = 5 \Rightarrow y = 4$

$2x - 3y + z = 7 \Rightarrow z = 15$

$U\bar{x} = \bar{y} \cdot \begin{bmatrix} 2 \\ 4 \\ 15 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 15 \end{bmatrix}$$

$z = 15/8$

$3y + z = 4 \Rightarrow 3y = 17/8$

$y = 17/24$

$$2x + 4y + 2z = 2$$

$$2x = 2 - 4(17/24) - 2(15/8)$$

$$= \frac{48}{24} - \frac{68}{24} - \frac{90}{24} = \frac{-110}{24}$$

$$x = \frac{-55}{24}$$

$$2x_1 + x_2 = 3$$
$$-x_1 + x_2 = 5$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow |A| = 3$$

$$b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix}}{|A|} = \frac{-2}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}}{|A|} = \frac{13}{3}$$

3. (a) (10 Points) Find an orthogonal matrix C such that

$$C^{-1}AC = \text{a diagonal matrix}, \ \text{for } A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

(b) (10 Points) Find the characteristic polynomial and the eigenvalues of

$$B = \begin{bmatrix} 3 & 0 & 0 & 13 \\ -25 & 7 & 11 & -6 \\ 18 & 0 & 1 & 5 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

$$|A-\lambda I| = (3-\lambda)^2 - 1 = (\lambda-2)(\lambda-4)$$

$\lambda=2$: $A-\lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow (A-\lambda I)\bar{v} = \bar{0} \Rightarrow \bar{v} = \begin{bmatrix} r \\ -r \end{bmatrix}$, so $\bar{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\lambda=4$: $A-\lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow (A-\lambda I)\bar{v} = \bar{0} \Rightarrow \bar{v} = \begin{bmatrix} s \\ s \end{bmatrix}$, so $\bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\Rightarrow \quad C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$|A-\lambda I| = \left| \begin{bmatrix} 3-\lambda & 0 & 0 & 13 \\ -25 & 7-\lambda & 11 & -6 \\ 18 & 0 & 1-\lambda & 5 \\ 0 & 0 & 0 & -2-\lambda \end{bmatrix} \right| = (7-\lambda) \cdot \begin{vmatrix} 3-\lambda & 0 & 13 \\ 18 & 1-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$= (7-\lambda)(-2-\lambda) \cdot \begin{vmatrix} 3-\lambda & 0 \\ 18 & 1-\lambda \end{vmatrix}$$

$$= (7-\lambda)(-2-\lambda)(3-\lambda)(1-\lambda)$$

$\Rightarrow$ eigenvalues: $\lambda = 7, -2, 3, 1$

4. (a) (10 Points) Find the least square solution to the following system

$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 + x_2 &= 0 \\ -x_2 &= 1 \\ x_1 &= 2 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(b) (10 Points) Let $<,>$ be the inner product on P_2 defined by

$$\langle P, Q \rangle = \int_0^1 P(x)Q(x)dx.$$

Let $P(x) = x - 1,\ Q(x) = x + 1$ and angle between $P(x)$ and $Q(x)$ be θ. Find $\cos\theta$.

$$A^t \cdot A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow (A^t A)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\Rightarrow \text{pinv}(A) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} = 1/3 \cdot \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\text{pinv}(A) \cdot \bar{b} = 1/3 \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} = 1/3 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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b) $$\langle p, q \rangle = \int_0^1 x^2 - 1\, dx = \frac{1}{3} - 1 = -2/3$$

$$\langle p, p \rangle = \int_0^1 (x-1)^2 dx = \int_{-1}^0 x^2 dx = 1/3 \Rightarrow \|p\| = 1/\sqrt{3}$$

$$\langle q, q \rangle = \int_0^1 (x+1)^2 dx = \int_1^2 x^2 dx = 7/3 \Rightarrow \|q\| = \sqrt{7/3}$$

so $$\cos\theta = \frac{-2/3}{1/\sqrt{3} \cdot \sqrt{7}/\sqrt{3}} = -2/\sqrt{7}$$

5. (a) (10 Points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y) = (x+y, x-2y)$. Find the inverse of T.

   (b) (10 Points) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$S(x,y) = (-6x+4y, -6x+3y, -3x+2y).$$

Let

$$\mathcal{B} = \{(1,2),(2,3)\},$$
$$\mathcal{B}' = \{(2,0,0),(0,-3,0),(0,0,1)\}$$

be bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Find the matrix representation A of T with respect to $\mathcal{B}$ and $\mathcal{B}'$.

$$T(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{-2-1} \cdot \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$T^{-1}(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{2}{3}x + \frac{1}{3}y, \frac{1}{3}x - \frac{1}{3}y\right)$$

b) $S(1,2) = (2,0,1) = (2,0,0) + 0\cdot(0,-3,0) + 1\cdot(0,0,1)$

$S(2,3) = (0,-3,0) = 0\cdot(2,0,0) + 1\cdot(0,-3,0) + 0\cdot(0,0,1)$

$$[S] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

6. (Consider the bases

$$\mathcal{B} = \{2, x+1, x^2\}$$
$$\mathcal{B}' = \{1, x, x+x^2\}$$

for P_2. Let $p = 5x^2 + 3x + 2$.

(a) (10 Points) Find $p_{\mathcal{B}}$ and $p_{\mathcal{B}'}$.

(b) (10 Points) Find the transition matrix P from $\mathcal{B}$ to $\mathcal{B}'$. Compute $p_{\mathcal{B}'}$ using P and $p_{\mathcal{B}}$.

$$5x^2 + 3x + 2 = -\tfrac{1}{2}\cdot 2 + 3(x+1) + 5\cdot x^2 \Rightarrow \quad p_{\mathcal{B}} = \begin{bmatrix} -1/2 \\ 3 \\ 5 \end{bmatrix}$$

$$5x^2 + 3x + 2 = 2\cdot 1 + -2x + 5(x^2+x) \Rightarrow \quad p_{\mathcal{B}'} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$$

$$\begin{array}{c|ccc} \mathcal{B}' \setminus \mathcal{B}: & 2 & x+1 & x^2 \\ \hline 1 & & & \\ x & & & \\ x^2+x & & & \end{array} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad p_{\mathcal{B}'} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \quad \checkmark$$