NAME: Solutions ☐ TTH ☐ MWF 10AM ☐ MWF 11AM

PLEDGE: ______________________________

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SIGNATURE: ______________________________

To get credit for a problem, you must show all of your reasoning and calculations. No calculators may be used.

Sabrina
Paramasamy
Paramasamy
Mike
Mike
Qian
London

Problem	Score
Multiple Choice	
1	
2	
3	
4	
5	
6	
Total	

Multiple Choice Answers

1	A
2	D
3	D
4	C
5	A
6	B
7	E
8	C
9	A
10	C
11	B
12	D
13	E
14	A
15	B
16	C

5 pts each, all or nothing

1. (a) (12 Points)Let

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & -2 & 3 \\ 4 & 8 & a \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ c \end{bmatrix}.$$

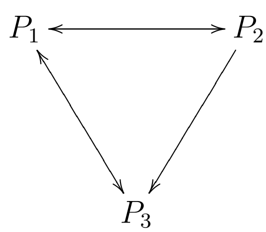
For what values of a and c does the system $A\mathbf{x} = \mathbf{b}$ have

(i) Many solutions?

(ii) A unique solution?

(iii) No solutions?

(b) (8 Points) Find the adjacency matrix A of the following communication network. Find the number of 2-paths from P_1 to P_2, from P_1 to P_3, and from P_2 to P_3 using the matrix A^2



$$\begin{array}{c} -2 \\ -4 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & 6 & 5 \\ 1 & -2 & 3 & 7 \\ 4 & 8 & a & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 8 & 0 & -9 \\ 0 & 16 & a-12 & c-28 \end{array}\right] -2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 8 & 0 & -9 \\ 0 & 0 & a-12 & c-10 \end{array}\right]$$

$a \neq 12 \Rightarrow$ unique solution

$a = 12$: $c \neq 10 \Rightarrow$ no solution

$c = 10 \rightarrow$ many solutions

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = A$$

$$A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

So 0 2-stage paths $P_1 \to P_2$

1 2-stage path $P_1 \to P_3$

1 2-stage path $P_2 \to P_3$

2. (a) (6 Points) Find an LU decomposition (if possible) of the matrix

$$C = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -1 & 5 & 9 \end{bmatrix}$$

(b) (10 Points) Solve the system $A\mathbf{x} = \mathbf{b}$ using the LU factorization $A = LU$;

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

(c) (4 Points) Using Cramer's rule solve the following system

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ -x_1 + x_2 &= 5 \end{aligned}$$

$$+1 \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -1 & 5 & 9 \end{bmatrix} -2 \Longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 7 & 6 \end{bmatrix} +7/3 \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 0 & 20 \end{bmatrix}$$

$$L: \quad \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & ? & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -7/3 & 1 \end{bmatrix}$$

$L\bar{y} = \bar{b}$

$$\begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$x = 2$

$\frac{1}{2}x + y = 5 \Rightarrow y = 4$

$2x - 3y + z = 7 \Rightarrow z = 15$

$U\bar{x} = \bar{y} \cdot \begin{bmatrix} 2 \\ 4 \\ 15 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 15 \end{bmatrix}$$

$z = 15/8$

$3y + z = 4 \Rightarrow 3y = 17/8$

$y = 17/24$

$$2x + 4y + 2z = 2$$
$$2x = 2 - 4(17/24) - 2(15/8)$$
$$= \frac{48}{24} - \frac{68}{24} - \frac{90}{24} = \frac{-110}{24}$$
$$x = \frac{-55}{24}$$

$$2x_1 + x_2 = 3$$
$$-x_1 + x_2 = 5$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow |A| = 3$$

$$b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix}}{|A|} = \frac{-2}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}}{|A|} = \frac{13}{3}$$

3. (a) (10 Points) Find an orthogonal matrix C such that

$$C^{-1}AC = \text{a diagonal matrix, for } A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

(b) (10 Points) Find the characteristic polynomial and the eigenvalues of

$$B = \begin{bmatrix} 3 & 0 & 0 & 13 \\ -25 & 7 & 11 & -6 \\ 18 & 0 & 1 & 5 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

$|A-\lambda I| = (3-\lambda)^2 - 1 = (\lambda-2)(\lambda-4)$ +2

$\lambda = 2$: $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow (A-\lambda I)\bar{v} = \bar{0} \Rightarrow \bar{v} = \begin{bmatrix} r \\ -r \end{bmatrix}$, so +2

$\bar{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ +1

$\lambda = 4$: $A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow (A-\lambda I)\bar{v} = \bar{0} \Rightarrow \bar{v} = \begin{bmatrix} s \\ s \end{bmatrix}$, so +2

$\bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ +1

$\Rightarrow C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ +2

(order of col doesn't matter. signs of col don't matter: one ±, the other same signs)

$|A-\lambda I| =$ +2 $\left| \begin{bmatrix} 3-\lambda & 0 & 0 & 13 \\ -25 & 7-\lambda & 11 & -6 \\ 18 & 0 & 1-\lambda & 5 \\ 0 & 0 & 0 & -2-\lambda \end{bmatrix} \right| = (7-\lambda) \cdot$ +2 $\begin{vmatrix} 3-\lambda & 0 & 13 \\ 18 & 1-\lambda & 5 \\ 0 & 0 & -2-\lambda \end{vmatrix}$

$= (7-\lambda)(-2-\lambda) \cdot$ +2 $\begin{vmatrix} 3-\lambda & 0 \\ 18 & 1-\lambda \end{vmatrix}$

$= (7-\lambda)(-2-\lambda)(3-\lambda)(1-\lambda)$ +2

$\Rightarrow$ eigenvalues: $\lambda = 7, -2, 3, 1$ +2

4. (a) (10 Points) Find the least square solution to the following system

$$\begin{aligned} x_1 + x_2 &= 4 \\ -x_1 + x_2 &= 0 \\ -x_2 &= 1 \\ x_1 &= 2 \end{aligned}$$

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(b) (10 Points) Let <, > be the inner product on P_2 defined by

$$\langle P, Q \rangle = \int_0^1 P(x)Q(x)dx.$$

Let $P(x) = x - 1$, $Q(x) = x + 1$ and angle between $P(x)$ and $Q(x)$ be θ. Find $\cos\theta$.

$$A^t \cdot A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow (A^t A)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \quad +3$$

$$\Rightarrow \operatorname{pinv}(A) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} = 1/3 \cdot \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \quad +3$$

$$+4 \quad \operatorname{pinv}(A) \cdot \bar{b} = 1/3 \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} = 1/3 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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b) $\langle p, q \rangle = \int_0^1 x^2 - 1 \, dx = \frac{1}{3} - 1 = -2/3$ $\quad +3$

$\langle p, p \rangle = \int_0^1 (x-1)^2 dx = \int_{-1}^0 x^2 dx = 1/3 \Rightarrow \|p\| = 1/\sqrt{3}$ $\quad +3$

$\langle q, q \rangle = \int_0^1 (x+1)^2 dx = \int_1^2 x^2 dx = 7/3 \Rightarrow \|q\| = \sqrt{7/3}$ $\quad +3$

so $\cos\theta = \dfrac{-2/3}{1/\sqrt{3} \cdot \sqrt{7}/\sqrt{3}} = -2/\sqrt{7}$ $\quad +1$

5. (a) (10 Points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + y, x - 2y)$. Find the inverse of T.

(b) (10 Points) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$S(x, y) = (-6x + 4y, -6x + 3y, -3x + 2y).$$

Let

$$\mathcal{B} = \{(1, 2), (2, 3)\},$$
$$\mathcal{B}' = \{(2, 0, 0), (0, -3, 0), (0, 0, 1)\}$$

be bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Find the matrix representation A of T with respect to $\mathcal{B}$ and $\mathcal{B}'$.

$$T(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad +3$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \frac{1}{-2-1} \cdot \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \quad +3$$

$$T^{-1}(x,y) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad +2$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{2}{3}x + \frac{1}{3}y, \frac{1}{3}x - \frac{1}{3}y\right) \quad +2$$

(if they get the answer and check: +10, even w/ no other work)

b) $S(1,2) = (2, 0, 1) = (2,0,0) + 0\cdot(0,-3,0) + 1\cdot(0,0,1)$ +3

$S(2,3) = (0, -3, 0) = 0\cdot(2,0,0) + 1\cdot(0,-3,0) + 0\cdot(0,0,1)$ +3

$$[S] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad +4$$

6. (Consider the bases

$$\mathcal{B} = \{2, x+1, x^2\}$$
$$\mathcal{B}' = \{1, x, x+x^2\}$$

for P_2. Let $p = 5x^2 + 3x + 2$.

(a) (10 Points) Find $p_{\mathcal{B}}$ and $p_{\mathcal{B}'}$.

(b) (10 Points) Find the transition matrix P from $\mathcal{B}$ to $\mathcal{B}'$. Compute $p_{\mathcal{B}'}$ using P and $p_{\mathcal{B}}$.

$$5x^2 + 3x + 2 = -\tfrac{1}{2}\cdot 2 + 3(x+1) + 5\cdot x^2 \Rightarrow P_{\mathcal{B}} = \begin{bmatrix} -1/2 \\ 3 \\ 5 \end{bmatrix} \quad +5$$

$$5x^2 + 3x + 2 = 2\cdot 1 + -2x + 5(x^2+x) \Rightarrow P_{\mathcal{B}'} = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \quad +5$$

$$\begin{array}{c|c} \mathcal{B}' \backslash \mathcal{B}: & \begin{array}{ccc} 2 & x+1 & x^2 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2+x \end{array} & \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad +7 \quad \Rightarrow \quad P_{\mathcal{B}'} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 3 \\ 5 \end{bmatrix} \quad +3$$

$$= \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} \quad \checkmark$$