Asymptotics in Homotopy Theory

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Outline

Definitions and History
- Goals of Algebraic Topology
- Homotopy Groups of Spheres

Computing Homotopy Groups
- Geometry and Cobordism
- Algebraic Model

Asymptotics in Stable Homotopy
- More Precise Identifications

Want to Understand Spaces

- Use algebra to distinguish spaces.
- Two parts:
  1. Find [computable] invariants of spaces
  2. Specify how to build spaces out of simpler ones.
- Example: [deRham] Cohomology, Handlebodies for manifolds, obstruction theory.
- Ideally, solutions to second part use invariants from the first.

Why Homotopy?

- "Continuous functions occur continuously"
- Elementary algebraic objects are discrete
- Passage to homotopy classes makes functions into discrete families.

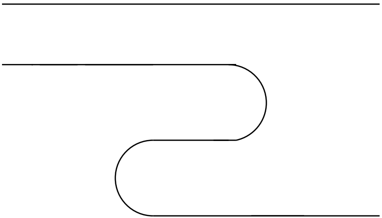
Definition

$f, g\colon X \to Y$ are homotopic if there is a map $F\colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

In other words, we can continuously deform f into g.

Picture!

Imagine 2 embeddings of the line into the plane:



If we pull the ends of the bottom string, then it looks like the top string.

Homotopy Groups

Definition

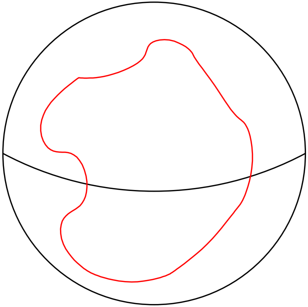
Let S^n be the collection of unit vectors in $\mathbb{R}^{n+1}$.
Let $\pi_n(X)$ be the collection of homotopy classes of maps $S^n \to X$.

- If $n = 1$, then this gives the *fundamental group* of X.
- If $n > 0$, then this is a group.
- If $n > 1$, then this group is commutative.

Measure ways to glue disks to X (up to homotopy).

Homotopy Groups of Spheres

- $\pi_n(S^m)$ describes how to attach disks to spheres.
- First step to build anything out of disks.
- If $n < m$, there is only one class of maps: Jordan Curve Theorem.



$\pi_n(S^m)$

$m \diagdown n$	1	2	3	4	5	6	7	8
1	$\mathbb{Z}$	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$\mathbb{Z}/2 \times \mathbb{Z}/2$

Few patterns:

- (Serre) $\pi_n(S^m)$ is almost always finite.
  At most two copies of $\mathbb{Z}$ each row.
- (Freudenthal) Diagonals stabilize. These are the *stable homotopy groups of spheres*.

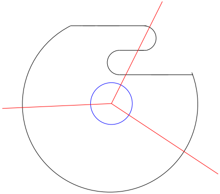
Definition

π_n^S is the stable group corresponding to $\pi_{n+k}S^k$.

Geometry

Very close ties between homotopy groups and geometry.

- Degree: Given $S^n \to S^n$, can count the number of preimages generically.



- Parity is an invariant ("detected in mod 2 homology")
- If we remember orientations, then get an integer invariant. This perfectly detects $S^n \to S^n$.

Cobordism

Can generalize this to other homotopy groups.

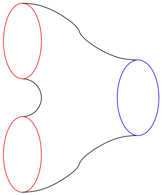
Definition

A framed manifold is a manifold embedded in $\mathbb{R}^N$ together with a basis for the normal vectors at each point.

Definition

Two manifolds M and N are cobordant if there is a W such that $\partial W = M \coprod N$.

Cobordism defines an equivalence relation on n-manifolds.



Homotopy Groups

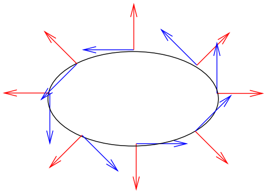
Theorem (Pontrjagin)

$\{$*Framed manifolds of dimension n embedded in* $\mathbb{R}^{k+n}\}$ *up to cobordism* $= \pi_{n+k} S^k$.

Idea: Pick a regular value of the map $S^{n+k} \to S^k$. It gets a framing from S^k. Pull that point back to get an n-dimensional manifold with a framing. Homotopy $\leftrightarrow$ Cobordism.

Theorem

$\pi_1^S = \mathbb{Z}/2$.



Filtrations

We need a way to isolate particular maps.
We use the length of factorizations invisible to a homology theory.
$f\colon X \to Y$ induces group homomorphism in homology $f_*\colon H_*(X) \to H_*(Y)$.

Definition

Given a map $f\colon X \to Y$, we say it has Adams filtration at least s if we can write it as a composite

$$X = X_0 \to \cdots \to X_s = Y,$$

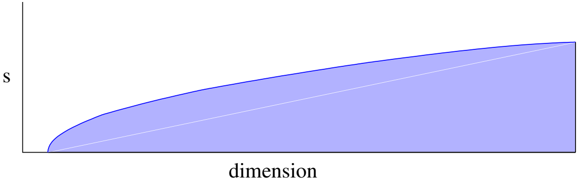
where each map induces the zero homomorphism in homology.

Example: The Hopf map $\eta\colon S^3 \to S^2$ has Adams filtration 1.

General Sketch of Homotopy Groups

Adams filtration lets us draw out a 2D region of all maps:

- vertical axis = s
- horizontal axis = topological dimension = n in π_n^S

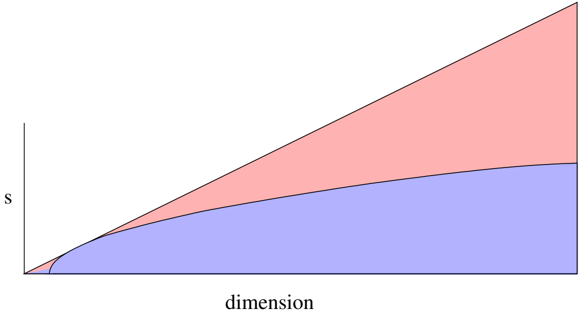


There is a vanishing curve $s = g(n)$

- Know $\lim \frac{g(n)}{n} = 0$
- Think that $g(n)$ looks like $\sqrt{n}$.

Algebraic Approximation

- Purely algebraic approximation: Adams-Novikov Spectral Sequence
- Built by considering all algebraically possible maps with the above filtration.



"Asymptotics" means the stuff in the red region.

Drawbacks

- Tricky to compute where there could be maps
- Very coarse approximation!

Two questions:

1. What's happening in the red region?
2. What does this mean for the region below the vanishing curve?

Work with Hopkins and Ravenel essentially answers the first question and provides a framework to answer the second.

Above the vanishing curve

- The entire algebraic story is governed by the theory of formal groups
- The algebraic approximation is the cohomology of the moduli stack of formal groups.
- Working one prime at a time, this story is controlled by the *height*: Chromatic Filtration.
- At each height, the stack is determined by a p-adic Lie group $\mathbb{G}_n$.
- $H^*(\mathbb{G}_n; \pi_* E_n)$ governs what happens above the vanishing line.

Morava Stabilizer Groups

- $\mathbb{G}_n$ is the automorphisms of any height n formal group law.
- $\mathbb{G}_n$ has finite virtual cohomological dimension.

$\Rightarrow$ Asymptotically, $H^*(\mathbb{G}_n; \pi_* E_n)$ is entirely controlled by finite subgroups.

- Example: $\mathbb{G}_1 = \mathbb{Z}_p^\times$, $\pi_* E_1 = \mathbb{Z}_p[u^{\pm 1}]$.
  If $p > 2$, this has finite cohomological dimension.
  If $p = 2$, $\mathbb{G}_1 = \mathbb{Z}_2 \times \mathbb{Z}/2$.
- If $p = 2$ and $G = \mathbb{Z}/2 \subset \mathbb{G}_n$, this is a completion of work of Kitchloo and Wilson.

Theorem (H.-Hopkins-Ravenel)

For finite G, $\pi_* E_n$ is an easily described G-algebra.

Essentially the symmetric algebra on the Dieudonné module.

Theorem (H.-Hopkins-Ravenel)

For $G = \mathbb{Z}/p$,

$$H^*_{Tate}(\mathbb{Z}/p; \pi_* E_n) = \mathbb{F}_{p^n}[\![\delta_1, \ldots, \delta_f]\!][\Delta^{\pm 1}][\beta^{\pm 1}] \otimes E(h_{1,0}, \ldots, h_{f,0}).$$

For finite G, $H^*(G; \pi_* E_n)$ cover the whole upper half plane.
Contains more and more of the red region.