Asymptotics in Homotopy Theory

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Outline

Definitions and History
- Goals of Algebraic Topology
- Homotopy Groups of Spheres

Computing Homotopy Groups
- Geometry and Cobordism
- Algebraic Model

Asymptotics in Stable Homotopy
- More Precise Identifications
Want to Understand Spaces

- Use algebra to distinguish spaces.
- Two parts:
  1. Find [computable] invariants of spaces
  2. Specify how to build spaces out of simpler ones.
- Example: [deRham] Cohomology, Handlebodies for manifolds, obstruction theory.
- Ideally, solutions to second part use invariants from the first.
Why Homotopy?

- “Continuous functions occur continuously”
- Elementary algebraic objects are discrete
- Passage to homotopy classes makes functions into discrete families.

**Definition**

\[ f, g : X \to Y \text{ are homotopic if there is a map } F : X \times I \to Y \text{ such that } F(x, 0) = f(x) \text{ and } F(x, 1) = g(x). \]

In other words, we can continuously deform \( f \) into \( g \).
Imagine 2 embeddings of the line into the plane:

If we pull the ends of the bottom string, then it looks like the top string.
Homotopy Groups

Definition
Let $S^n$ be the collection of unit vectors in $\mathbb{R}^{n+1}$. Let $\pi_n(X)$ be the collection of homotopy classes of maps $S^n \to X$.

- If $n = 1$, then this gives the fundamental group of $X$.
- If $n > 0$, then this is a group.
- If $n > 1$, then this group is commutative.

Measure ways to glue disks to $X$ (up to homotopy).
Homotopy Groups of Spheres

- $\pi_n(S^m)$ describes how to attach disks to spheres.
- First step to build anything out of disks.
- If $n < m$, there is only one class of maps: Jordan Curve Theorem.
Computing Homotopy Groups

Asymptotics in Stable Homotopy

### Definitions and History

#### Computing Homotopy Groups

\[ \pi_n(S^m) \]

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#### Few patterns:

- (Serre) \( \pi_n(S^m) \) is almost always finite.
  At most two copies of \( \mathbb{Z} \) each row.

- (Freudenthal) Diagonals stabilize. These are the stable homotopy groups of spheres.

### Definition

\( \pi_n^S \) is the stable group corresponding to \( \pi_{n+k} S^k \).
Geometry

Very close ties between homotopy groups and geometry.

- Degree: Given $S^n \to S^n$, can count the number of preimages generically.

- Parity is an invariant (“detected in mod 2 homology”)
- If we remember orientations, then get an integer invariant. This perfectly detects $S^n \to S^n$. 

\[ \text{Diagram of a circle with arrows, indicating some topological structure.} \]
Cobordism

Can generalize this to other homotopy groups.

Definition
A framed manifold is a manifold embedded in $\mathbb{R}^N$ together with a basis for the normal vectors at each point.

Definition
Two manifolds $M$ and $N$ are cobordant if there is a $W$ such that $\partial W = M \sqcup N$.

Cobordism defines an equivalence relation on $n$-manifolds.
Homotopy Groups

Theorem (Pontrjagin)

\{ Framed manifolds of dimension \(n\) embedded in \(\mathbb{R}^{k+n}\) \} up to cobordism = \(\pi_{n+k} S^k\).

Idea: Pick a regular value of the map \(S^{n+k} \to S^k\). It gets a framing from \(S^k\). Pull that point back to get an \(n\)-dimensional manifold with a framing. Homotopy \(\leftrightarrow\) Cobordism.

Theorem

\(\pi^S_1 = \mathbb{Z}/2\).
Filtrations

We need a way to isolate particular maps. We use the length of factorizations invisible to a homology theory. $f: X \to Y$ induces group homomorphism in homology $f_*: H_*(X) \to H_*(Y)$.

**Definition**

*Given a map $f: X \to Y$, we say it has Adams filtration at least $s$ if we can write it as a composite*

$$X = X_0 \to \cdots \to X_s = Y,$$

*where each map induces the zero homomorphism in homology.*

**Example:** The Hopf map $\eta: S^3 \to S^2$ has Adams filtration 1.
General Sketch of Homotopy Groups

Adams filtration lets us draw out a 2D region of all maps:

- vertical axis = $s$
- horizontal axis = topological dimension = $n$ in $\pi_n^S$

There is a vanishing curve $s = g(n)$

- Know $\lim_{n \to \infty} \frac{g(n)}{n} = 0$
- Think that $g(n)$ looks like $\sqrt{n}$. 
Algebraic Approximation

- Purely algebraic approximation: Adams-Novikov Spectral Sequence
- Built by considering all algebraically possible maps with the above filtration.

“Asymptotics” means the stuff in the red region.
**Drawbacks**

- Tricky to compute where there could be maps
- Very coarse approximation!

Two questions:

1. What’s happening in the red region?
2. What does this mean for the region below the vanishing curve?

Work with Hopkins and Ravenel essentially answers the first question and provides a framework to answer the second.
Above the vanishing curve

- The entire algebraic story is governed by the theory of formal groups.
- The algebraic approximation is the cohomology of the moduli stack of formal groups.
- Working one prime at a time, this story is controlled by the height: Chromatic Filtration.
- At each height, the stack is determined by a $p$-adic Lie group $\mathbb{G}_n$.
- $H^*(\mathbb{G}_n; \pi_* E_n)$ governs what happens above the vanishing line.
Morava Stabilizer Groups

- \( \mathbb{G}_n \) is the automorphisms of any height \( n \) formal group law.
- \( \mathbb{G}_n \) has finite virtual cohomological dimension.

\[ \Rightarrow \quad \text{Asymptotically, } H^*(\mathbb{G}_n; \pi_* E_n) \text{ is entirely controlled by finite subgroups.} \]

- Example: \( \mathbb{G}_1 = \mathbb{Z}_p^\times, \pi_* E_1 = \mathbb{Z}_p[u^{\pm 1}] \).
  If \( p > 2 \), this has finite cohomological dimension.
  If \( p = 2 \), \( \mathbb{G}_1 = \mathbb{Z}_2 \times \mathbb{Z}/2 \).

- If \( p = 2 \) and \( G = \mathbb{Z}/2 \subset \mathbb{G}_n \), this is a completion of work of Kitchloo and Wilson.
Theorem (H.-Hopkins-Ravenel)

For finite $G$, $\pi_*E_n$ is an easily described $G$-algebra.
Essentially the symmetric algebra on the Dieudonné module.

Theorem (H.-Hopkins-Ravenel)

For $G = \mathbb{Z}/p$,

\[
H^*_{\text{Tate}}(\mathbb{Z}/p; \pi_*E_n) = \mathbb{F}_p^n \left[ \delta_1, \ldots, \delta_f \right][\Delta^{\pm 1}][\beta^{\pm 1}] \otimes E(h_{1,0}, \ldots, h_{f,0}).
\]

For finite $G$, $H^*(G; \pi_*E_n)$ cover the whole upper half plane. Contains more and more of the red region.