Thank you for the invitation to speak. One of the first papers that I read in equivariant homotopy was Namboodiri's on equivariant vector fields on spheres. Meditating on this paper has shaped significantly my understanding and thinking. So this is a true honor.

I'm going to talk about what we might mean by "even" in the equivariant context. I'll start with a classical story, then focus on newer, equivariant versions.

I'm going to assume we all know what "even" means in $\mathbb{Z}$.

In spaces, we have 2 notions of "evenness":

① having only even cells – this is "presentation dependent"

② having only even homotopy groups

Some of the most geometrically meaningful spaces have one or both of these.

Ex: $\mathbb{C}P^n$ has a cell structure with only even cells $\forall n$ & $\mathbb{C}P^\infty$ also has only even htpy

Building on this example, Schubert cells $\Rightarrow$ $Gr_k(\mathbb{F}^n)$ has a cell decomposition w/ cells $\mathbb{F}^i$.

In particular, $Gr_k(\mathbb{C}^n)$ has only even cells. Same is true as $n \to \infty$. The bi-infinite limit

$\varinjlim_k \varinjlim_n Gr_k(\mathbb{C}^n) \simeq BU$ also has only even homotopy. Moreover, this is "natural" in the field.

These two contrasting notions of "even" cover some of the most important ideas in algebraic top.

Def An even periodic cohomology theory E is one for which $E^0(S^1) = 0$ & $E^0(S^2) \cong E^0(S^0)$

The prototype is complex K-theory: $K^0(S^1) = 0$ & $K^0(S^2) \cong K^0(S^0)$ is Bott Periodicity

Prop If E is even periodic, then $E^*(\underbrace{\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty}_{n}) \cong E^*(pt)[[x_1, \dots, x_n]]$ $\quad |x_i| = 0$

← only has even cells!

The multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{-\otimes-} \mathbb{C}P^\infty$ induces

$$E^*(pt)[[x]] \longrightarrow E^*(pt)[[y,z]], \quad x \longmapsto f(y,z)$$

a formal group law. ↙ this is the most "important object in alg top"

If X has even cells, then same thing is true: AHSS always collapses!

Moral: Easy to map out of spaces w/ even cells (& into spaces w/ even htpy)

Def A space X is even if it is s.c. & $\forall n \geq 0$

① $H_{2n+1}(X) = 0$ & $H_{2n}(X)$ free & f.g & ② $\pi_{2n+1} X = 0$. $\Rightarrow$ $\pi_{2n} X$ is torsion free

① Can be reformulated via Universal Coefs: $H^{2n+1}(X;M)=0$ for all M & X finite type. (or for $\mathbb{Z}$!)

These form an exceptionally nice class of spaces! In particular, they are quite algebraic.

Prop If $X = A\cup(\text{even cells})$ and Y is even, then any map $A\to Y$ extends to $X\to Y$.

Pf: Obstructions are in $H^{n+1}(X,A;\pi_n Y)$. □

Cor If X is even, then X is an H-space:
$$\begin{tikzcd} X\times X \arrow[r, "m"] & X \\ X\vee X \arrow[u] \arrow[ur, "\nabla"'] & \end{tikzcd}$$
In fact, all $\simeq$ ∞-loop spaces

Cor If X is even & A is a retract, then A is even & $X\simeq A\times F$ with F even. i.e. "there is no Ext"

In particular, we can form a prime decomp.

"Def" Let Y_{2k} be the space formed by maximally efficiently killing the odd htpy of S^{2k}.

i.e. choose minimal sets of generators for $\pi_{2n-1}(Y_{2k}^{2n-2})$ and cone them off.

Thm (Priddy-Wilson) This is a well-defined htpy type.

Aside: Can also start w/ $K(\mathbb{Z},2k)$ and kill odd cohomology.

The spaces Y_{2k} are the "primes" for even spaces.

Thm (Wilson) Y_{2k} is irreducible $\forall k$ & any even space is a product of Ys.

Now the real surprise!

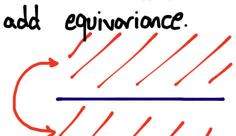
MU is $\mathbb{C}$ bordism. Plays a fundamental role in alg top.

Thm (Wilson) For all n, the connected components of $\underline{MU}_{2n} := \Omega^\infty(S^{2n}\wedge MU)$ are even.

Cor If X is even, then $\pi_{2k}X$ is torsion free.

So in fact, the story for $\mathbb{CP}^\infty$, BU, etc was generic! These have cells of the form $\mathbb{C}^n$.

Now add equivariance. (action by a fixed G.) Here $\mathrm{Gal}_{\mathbb{R}}(\mathbb{C})$ acts coordinatewise, w/ fixed points $\mathbb{RP}^\infty$, etc.



$\mathbb{R}$ fixed. In this case, "even cells" are naturally copies of $\mathbb{C}^n \cong n\rho_2$. Say more on Monday.

Basic Idea Replace "2" with "ρ" and essentially everything goes through.

Thm (H.-Hopkins) For all n, $\underline{MU\mathbb{R}}_{n\rho} := \Omega^{\infty\rho}(S^{n\rho}\wedge MU_{\mathbb{R}})$ has even cells.

What's a consequence of this? One thing is that the unstable Adams-Novikov SS for spaces with even cells is much simpler than expected. Here we resolve by $\Omega^\infty(MU\wedge\Sigma^\infty(-))$. If X is even then so is $\Omega^\infty(MU\wedge\Sigma^\infty X)$ $\Rightarrow$ unstable ASS is extra computable!

Conj (Asok-Hopkins) This works motivically to let us compute $\mathrm{Vect}^n(X)=[X,BGL_n]_{\mathbb{A}^1}$ for X affine even, etc.