Thank you for the invitation to speak. One of the first papers that I read in equivariant homotopy was Nambuichi’s on equivariant vector fields on spheres. Meditating on this paper has shaped significantly my understanding and thinking. So this is a true honor.

I’m going to talk about what we might mean by “even” in the equivariant context. I’ll start with a classical story, then focus on newer, equivariant versions.

I’m going to assume we all know what “even” means in \( \mathbb{Z} \).

In spaces, we have 2 notions of “evenness”:

1. having only even cells - this is “presentation dependent”
2. having only even homotopy groups

Some of the most geometrically meaningful spaces have one or both of these:

Example: \( \mathbb{C}P^n \) has a cell structure with only even cells. \( \mathbb{H}n \neq \mathbb{C}P^n \) also has only even homotopy.

Building on this example, Schubert cells in \( \text{Gr}_k(\mathbb{F}^n) \) have a cell decomposition w/ cells \( \mathbb{F}^i \).

In particular, \( \text{Gr}_k(\mathbb{C}^n) \) has only even cells. Same is true as \( n \to \infty \). The “infinite limit”

\[
\lim_{n \to \infty} \lim_{k \to \infty} \text{Gr}_k(\mathbb{C}^n) = BU
\]

also has only even homotopy. Moreover, this is “natural” in the field.

These two contrasting notions of “even” cover some of the most important ideas in algebraic topology.

**Def.** An even periodic cohomology theory \( E \) is one for which \( E^*(S^1) = 0 \neq E^0(S^1) \).

The prototype is complex K-theory: \( K^*(S^1) = 0 \neq K^0(S^1) \approx K^0(S) \) is Bott Periodicity.

**Prop.** If \( E \) is even periodic, then \( E^*(\mathbb{C}P^\infty) = E^*(pt) \cup [x] \), \( n \) only has even cells!

The multiplication map \( \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \) induces:

\[
E^*(pt) \cup [x] \longrightarrow E^*(pt) \cup [y, z], \quad \text{a formal group law.}
\]

If \( X \) has even cells, then something else is true: AHSS always collapses!

**Moral:** Easy to map out of spaces w/ even cells (i.e., into spaces w/ even homotopy).

**Def.** A space \( X \) is even if it is s.c. \( \forall n \geq 0 \)

- \( H_{ev}(X) = 0 \neq H_{ev}(X) \) free (if \( y \neq 0 \))
- \( \pi_{2n}(X) = 0 \Rightarrow \pi_{2n}X \) is torsion free.
Can be rephrased via Universal Coeff: \( H^{\infty}(x; M) = 0 \) for all \( M \neq X \) finite type.

These form an exceptionally nice class of spaces! In particular, they are quite algebraic.

**Prop.** If \( X = A \vee (\text{even cells}) \) and \( Y \) is even, then any map \( A \to Y \) extends to \( X \to Y \).

**Thm:** Obstructions are in \( H^{\infty}(x; A \vee \Sigma Y) \).

In fact, all \( \infty \)-loop spaces are \( X \).

**Cor.** If \( X \) is even, then \( X \) is an \( H \)-space: \( X \times X \to X \).

**Cor.** If \( X \) is even \( \Rightarrow \) \( \text{A is a retract} \Rightarrow \text{A is even} \) \( \Rightarrow X \simeq A \vee F \) with \( F \) even.

In particular, we can form a prime decap.

**Def.** Let \( Y_{2h} \) be the space formed by maximally efficiently killing the odd htpy of \( S^{2h} \).

I.e. choose minimal sets of generators for \( \pi_{2h-1}(Y_{2h}) \) and cone them off.

**Thm.** (Paddy-Wilson) This is a well-defined htpy type.

Aside: Can also start w/ \( KS(2; A) \) and kill odd cohomology.

The spaces \( Y_{2h} \) are the `primes` for even spaces.

**Thm.** (Wilson) \( Y_{2h} \) is irreducible \( \forall h \) \( \Rightarrow \) any even space is a product of \( Y_{2h} \).

Now the real surprise!

**Thm.** (Wilson) For all \( n \), the connected components of \( MU_{2n} = \Sigma^{2n} (S^{2n} \wedge MU) \) are even.

**Cor.** If \( X \) is even, then \( \pi_{2h} X \) is torsion free.

So in fact, the story for \( CP^\infty \), \( BU \), etc was generic! These have cells of the form \( C^n \).

**Thm.** (H.-Hopkins) For all \( n \), \( MU_{2p} \simeq S^{2p} \wedge (S^{2p} \wedge MU) \) has even cells.

What's a consequence of this? One thing is that the unstable Adams-Novikov SS for spaces with even cells is much simpler than expected. Here we resolve by \( S^{2n} \wedge (\Sigma^\infty MU \wedge \Sigma^\infty X) \). If \( X \) is even, then so is \( S^{2n} \wedge (\Sigma^\infty X) \Rightarrow \text{unstable ASS is extra computable}.\)

**Conj.** (Asok-Hopkins) This works motivically to let us compute \( \text{Vect}^h(X) = [X, BGL_{2n}]_A \) for \( X \) affine even, etc.