Goal: understand equivariant commutative ring spectra. Have an equivariant spectrum R + a homotopy coherent comm. multiplication + homotopy coherent norms.

What extra structure do we have and what does this mean for G-categories in general?

Step back: what is this norm thing?!

In genuine G-spectra: $\mathcal{S}_G$: objects = spectra w/ G-action, morph: G-space/spectrum of maps w/ conj. action. $\Big\}$ I a G-set, $\bigvee_I X_i \xrightarrow{\simeq} \prod_I X_i$

slight gen of $\otimes$ op.

trivial I: naive spectra
(not named)
all fin. I: genuine G-spectra

Slightly different from the usual definition: $S^V \wedge S^{-V} \xrightarrow{\simeq} S^0$ for some V. Lewis showed this has the same information (he did so via the transfer)

$\mathcal{S}_G$ is a closed sym. monoidal category w/ $\wedge$ & have Comm_G = cat of G-com. ring spectra. Again, the homs are G-spaces.

In particular, these are tensored over G-sets/spaces: $G\text{-Comm}(X \otimes T, R) = \mathrm{Top}(X, \mathrm{Comm}(T, R))$.

<u>Thm</u> (H.-Hopkins-Ravenel) The tensoring operation extends to $-\otimes - : \mathcal{S}et_G^{\mathrm{fin,iso}} \times \mathcal{S}_G \to \mathcal{S}_G$, sym. monoidal in both factors.

For G/H, $G/H \otimes X = N_H^G(i_H^* X)$, $N_H^G : \mathcal{S}_H \to \mathcal{S}_G$ is the norm used often in the Kervaire proof.

① N_H^G is sym. monoidal

② $N_H^G(S^V) = S^{\mathrm{Ind}_H^G V}$, $V \in RO(H)$.

Have this in algebra too: M an H-module, can form $M^{\otimes G/H} =: N_H^G M = \underbrace{M \otimes \cdots \otimes M}_{G/H}$

What is the G-action? $f: G \to \Sigma_{G/H}$, and let $\Gamma_f \subseteq G \times \Sigma_{G/H}$ be the graph.

Then $\boxed{\mathbb{Z}\left[{}^{G \times \Sigma_{G/H}}/_{\Gamma_f}\right] \otimes_{\mathbb{Z}[\Sigma_{G/H}]} M^{\otimes |G/H|} =: M^{\otimes G/H}.}$

Same is true in spectra: $\boxed{N_H^G X \simeq (G \times \Sigma_{G/H}/\Gamma)_+ \wedge_{\Sigma_{G/H}} X^{\wedge |G/H|}.}$

Thus we have <u>more</u> structure on $\mathcal{S}_G$ than just sym. monoidal. $\mathcal{S}_G$ is "G-symmetric monoidal" (We can smash over a G-set, not just a set)

This actually makes something intuitive, so we can talk about commutative rings again:

Let $\mathcal{F}$ be a cofamily of subgroups ($H \in \mathcal{F}$, $K \supseteq gHg^{-1} \Rightarrow K \in \mathcal{F}$). Then we have a cat $\mathcal{S}et_G^{\mathcal{F}}$: obj = $\{X \mid \mathrm{stab}(x) \in \mathcal{F} \ \forall x \in X\}$.

<u>Def</u>: If $\mathcal{C}$ is G-sym. monoidal, then M is an $\mathcal{F}$-com. monoid iff $- \otimes M$ extends to a functor $\mathcal{S}et_G^{\mathcal{F}} \to \mathcal{C}$.

Well... If $\mathcal{C}$ is $\mathcal{S}_G$, Top, this is always genuinely commutative! We need a homotopical version, and I will come back to this.

<u>Question</u>: When does Bousfield localization preserve ($\mathcal{F}$)-commutative?

(H.-Hopkins) When the category of acyclics is closed under $X \otimes -$ for $X \in \mathcal{S}et_G^{\mathcal{F}}$.

This is a different version of the result than I've discussed before, but the difference is merely expository.

E a G-spectrum, $\mathcal{Z}_E$ = cat of E-acyclics, $G = C_4$

$\mathcal{Z}_E$ closed under:	$\wedge$	$\wedge, N_{C_2}^{C_4}$	$\wedge, N_{C_2}^{C_4}, N_e^{C_4}$
$L_E(R)$ is...	coherently comm.	coherently comm + coherent $N_{C_2}^{C_4}$	commutative

Let U be a G-universe, then $\mathcal{L}_n(U) = \mathcal{L}(U^n, U)$ is an operad in G-spaces, underlain by an E_∞-operad. Algebras over this are the homotopical versions. Have no preferred product or norms, etc.

Mandell tells us that if R is an E_∞-ring spectrum, then the category of modules is symmetric monoidal. So a "dual" to the category of acyclics gives us the structure of the cat of modules: If R is a commutative $\mathcal{F}$-algebra, then the category of modules has $-\otimes-: \mathcal{S}et_G^{\mathcal{F}} \times R\text{-Mod} \to R\text{-Mod}$. If R is commutative G-spectrum, then R-Mod has norms: ① $N_H^G \iota_H^* M$ is naturally a $N_H^G \iota_H^* R$-mod

② Base-change. Aside: this gives a relative THH!

Moral: If R isn't a comm. ring spectrum, then we are just losing extra structure on R-Mod.

Algebra: Two candidates: G-Mod, G-Mackey. Both are "genuine" equivariant cats, auto-enriched, G-symmetric monoidal:

G-Mod: $N_H^G \iota_H^* M =$ form. above

G-Mackey: ⓐ $\underline{M} \mapsto H\underline{M} \mapsto NH\underline{M} \mapsto N\underline{M} = \underline{\pi}_0 N\underline{M}$.

ⓑ (Mazur's thesis)

G-Mod has a similar draw-back to spectra: a com. ring obj is automatically G-comm. This is just too weak.

Mackey_G has more flexibility. In particular, we have examples of $\mathcal{F}$-algebras:

$$\underline{I}^* = \begin{tikzcd} \mathbb{Z}\oplus\mathbb{Z} \arrow[d, bend right] \\ \mathbb{Z} \arrow[u, bend right] \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$$

(the dual to the augmentation ideal). This is $\underline{\pi}_0 S^\lambda = \underline{\pi}_0 S^{\infty\lambda}$ where $\lambda =$ defining rep.

① $\underline{I}^*$ is a comm. ring ($S^{\infty\lambda}$ is homotopy com)

② $N_{C_2}^{C_4} \iota_{C_2}^* \underline{I} = \underline{I}$ ($\iota_{C_2}^* S^{\infty\lambda} = S^{\infty\sigma}$ & $\text{Ind}_{C_2}^{C_4} \sigma = \lambda$)

③ $N_e^{C_4} \iota_e^* \underline{I} = 0$.

} $\underline{I}$ is a $\mathcal{F}$-algebra, $\mathcal{F} = \{C_2, C_4\} = \{C_4\}$

The identical computation produces a family of examples, one for any cofamily in C_{p^n}.

There is a special case: $\mathcal{F} = \text{All}$. $\mathcal{F}$-algebras have another name in the literature, Tambara functors. So we get extra structure on the category of modules over a Tambara functor, namely norms.

Still missing something. $\boxtimes$, the monoidal product in G-Mackey is the Kan extension of $\otimes$ in Mod over $\times$ in G-sets. Both Mod & $\mathcal{S}et_G$ are G-symmetric monoidal. So we should just left Kan extend in the "right" G-way, but I don't yet know how.